Logarithmic nonlinear Schrödinger equation in theory of quantum Bose liquids

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I. Logarithmic Schrödinger equation: *Origins*

LogSE can be introduced independently of the theory of quantum fluids, in different ways.

Way 1.

Consider the generic $U(1)$-symm NLSE [Bialynicky-Birula&Mycielski'75]:

$$\left[-i\hbar\,\partial_t - \frac{\hbar^2}{2m}\vec{\nabla}^2 + V_{\rm ext}(\vec{x},\, t) + F(|\Psi|^2)\right]\Psi = 0$$

and ask the question: at which *F* this equation obeys the separability property, namely:

(a) Solution for a composite system is a product of the solutions for mutually <u>independent</u> subsystems

(b) Energy of a product state is a sum of energies of its components:

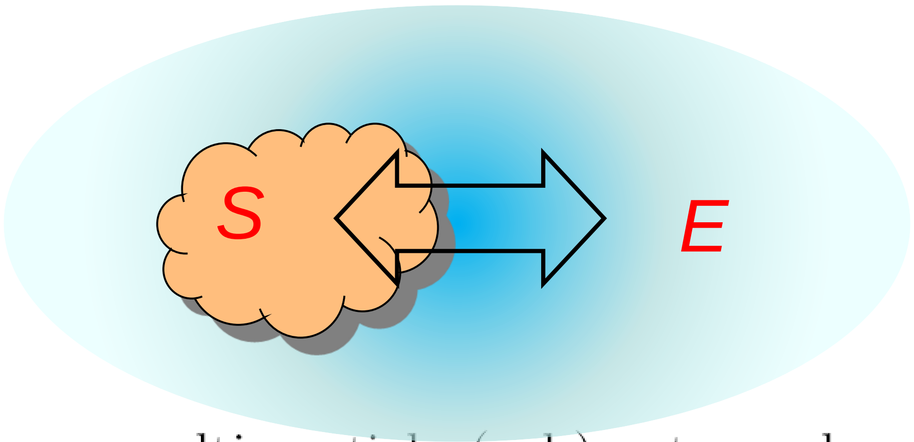
$$E(\Psi_1\Psi_2) = E(\Psi_1) + E(\Psi_2)$$

One answer *F* = 0 (linear SE) but this is not the most general one. Due to its properties – there comes logarithm.

$$\left[-i\hbar\,\partial_t - \frac{\hbar^2}{2m}\vec{\nabla}^2 + V_{\rm ext}(\vec{x},\, t) - \beta^{-1}\ln\left(a^3|\Psi|^2\right)\right]\Psi = 0$$

Way 2.

Based on arguments which come from open quantum systems and quantum information theory [Yasue'78, Brasher'91].



Let us consider a multi-particle (sub)system whose dynamics is described by the Hamiltonian-type operator $\hat{\mathbf{H}}$. Besides, this subsystem is in a contact with its environment such that there is an exchange of energy and information. The state of the system is described by the vector $|\Psi\rangle$.

If the Hamiltonian does not depend on wave function then in the Schrödinger coordinate representation we recover the linear differential equation for Ψ.

However, in general the interactions between the particles comprising the subsystem depend on the distribution $|\Psi|^2$ of the particles in the configuration space. To determine this distribution, i.e., to extract, transfer and store the information in a particular configuration of matter, one requires certain amount of energy per bit, call it ε. The information acquired upon measurement of the state is proportional to the logarithm of the probability of an outcome Ψ,

$$I_\Psi = -\log_2(\Xi|\Psi|^2) = -\ln(\Xi|\Psi|^2)/\ln 2$$

Hence the contribution to the Hamiltonian [Brasher'91]:

$$\hat{\mathbf{H}} \to \hat{\mathbf{H}}' = \hat{\mathbf{H}} - \varepsilon \log_2(\Xi|\Psi|^2)$$

Associated quantum entropy (min on delta distributions, max on uniform =>measures quantum spreading) [Everett'57]:

$$S_\Psi = -k_B \langle\Psi| \ln(\Xi|\Psi|^2)|\Psi\rangle$$

Conjugated temperature which can be formally associated with our entropy:

$$T_\Psi \equiv (k_B \beta)^{-1} = (\partial E' / \partial S_\Psi)_\Omega = \varepsilon / (k_B \ln 2)$$

where the total energy of the system can be written as:

$$E' = \langle \Psi | \hat{\mathbf{H}}' | \Psi \rangle = E + T_\Psi S_\Psi$$

$$\hat{\mathbf{H}}' = \hat{\mathbf{H}} - \varepsilon \log_2(\Xi |\Psi|^2)$$

$$S_\Psi = -k_B \langle \Psi | \ln (\Xi |\Psi|^2) | \Psi \rangle$$

whereas the "free energy":

$$E = \langle \Psi | \hat{\mathbf{H}} | \Psi \rangle = E' - T_\Psi S_\Psi$$

The last term is the energy engaged in handling the information and thus unavailable to do dynamical work. Replacing ε with β we arrive at LogSE in our notations:

$$\left[\hat{\mathbf{H}} - \beta^{-1} \ln(\Omega |\Psi|^2) \right] \Psi = E' \Psi$$

Environment can induce non-linearities [Gisin, Kostin, Yasue, Breuer & Petruccione].

If the effect of environment is purely "informational" – we obtain LogSE

$$I_\Psi = -\log_2(\Xi|\Psi|^2) = -\ln(\Xi|\Psi|^2)/\ln 2$$

<u>Problem</u>: E. in general can be a very complex object (thermal effects, noise, particle source/sink, etc.) hence in which physical systems the "informational" contribution becomes dominant (or detectable, at least)? Perhaps: $T \to 0$, many-body systems, superfluids…

Separate problem: Fundamental physics behind LogSE ?

$$E' = \langle \Psi | \hat{\mathbf{H}}' | \Psi \rangle = E + T_\Psi S_\Psi$$

$$\boxed{S_\Psi = -k_B \langle \Psi | \ln\left(\Xi|\Psi|^2\right) | \Psi \rangle}$$

Everett–Hirschmann uncertainty relation:

Special case of the Sobolev inequality.
It has been shown that this uncertainty relation is stronger than the Heisenberg one, full bibliography can be found in [Avdeenkov & KZ'11].

Other ways of deriving/introducing LogSE

Way 3.
Dilatation/conformal covariance (in relativistic field theory) [Rosen'69].

Way 4.
Principle of the maximal information (Everett) entropy.

… and so on…

Many things are still to be understood…

Ground state

$$\frac{\hbar^2}{2m}\Delta\Psi + \left[\mu + \beta^{-1}\ln\left(a^3\Psi^2\right)\right]\Psi = 0$$

$$\int n d^3x = 4\pi\int_0^\infty n(r)dr = N$$

$$\Psi = \sqrt{n(r)}$$

If $\beta > 0$ - the Gaussian-shaped solution:

$$n(r) = \Psi^2 = \frac{N}{\pi^{3/2}a_\beta^3}e^{-(r/a_\beta)^2}$$

self-sustainable,
localized,
stable (V-K)

$$\mu = 3\beta^{-1}\left[1 - \ln\left(\frac{N^{1/3}}{\sqrt{\pi}}\frac{a}{a_\beta}\right)\right] = -\beta^{-1}\ln\left(N/N_0\right)$$

where:

$$N_0 \equiv \left(e\sqrt{\pi}a_\beta/a\right)^3$$

$$a_\beta = \sqrt{A_\beta} = \hbar\sqrt{\beta/2m}$$

The total energy approaches minimum on this solution – this can be proven by using Everett–Hirschmann(-Sobolev) inequality [Bialynicki-Birula&Mycielski'76].

Summary: the most important properties of LogSE:

- Separability of non-interacting subsystems (as in the linear theory): the solution of the LogSE for the composite system is a product of solutions for the uncorrelated subsystems;
- Energy is additive for non-interacting subsystems (as in linear theory);
- Planck relation holds (as in the linear theory);
- All symmetry properties of the many-body wave-functions with respect to permutations of the coordinates of identical particles are preserved in time (as in the linear theory);
- Superposition principle is relaxed to the weak one: the sum of solutions with negligible overlap is also a solution;
- Free-particle solutions, called gaussons, have the coherent-states form, and upon the Galilean boost they become the uniformly moving Gaussian wave packets modulated by the de Broglie plane waves;
- Expressions for the probability density and current are the same as in the linear theory.

Thus, LogSE is the “minimally nonlinear” NL SE: among all NLSEqs, it preserves largest amount of properties of the conventional (linear) SE

II. Logarithmic quantum Bose liquid: General theory

A. V. Avdeenkov and K.G. Zloshchastiev,
J. Phys. B: At. Mol. Opt. Phys. **44** (2011) 195303 [arXiv:1108.0847]

Prologue: Gross-Pitaevskii equation

$$i\hbar \frac{\partial}{\partial t}\psi(t,\mathbf{x}) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\rm ext}(\mathbf{x}) + \lambda\, |\psi(t,\mathbf{x})|^2\right)\psi(t,\mathbf{x})$$

When derived in a theory of BEC, Gross-Pitaevskii (a.k.a. cubic Schrödinger) equation is a result of few approximations:

1) Mean field (HF)
2) Neglect excited states, only s-wave (BEC condition)
3) Neglect multi-body (3+) interactions
4) 2-body interaction is short-range (delta-like)
5) If perturbation theory - neglect anomalous contributions to self-energy

For diluted cold gases the GP approximation is sufficient (but even there it doesn't work in some cases, e.g., low-D).

In general: needed higher terms or even infinite series involving all powers of $|\psi|^2$ [Belyaev'58, Schick'71].

This is where non-polynomial wave equations come for rescue…

Let's consider quantum liquid (strongly-interacting BEC) whose wf is governed not by the GP equation,

$$i\hbar \frac{\partial}{\partial t}\psi(t,\mathbf{x}) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{x}) + \lambda\ |\psi(t,\mathbf{x})|^2\right)\psi(t,\mathbf{x})$$

but by the logarithmic one, defined as:

$$\left[-i\hbar\,\partial_t - \frac{\hbar^2}{2m}\vec{\nabla}^2 + V_{\text{ext}}(\vec{x},\,t) - \beta^{-1}\ln\left(a^3|\Psi|^2\right)\right]\Psi = 0$$

What properties would the corresponding liquid have?

Compare Log-BEC and GP-BEC?

Basic comparison

GP vs Logarithmic BEC:

$$\left[-i\hbar\,\partial_t - \frac{\hbar^2}{2m}\vec{\nabla}^2 + V_{\text{ext}}(\vec{x},\ t) + F(|\Psi|^2)\right]\Psi = 0$$

Definition:

Particle density of BEC:

$$\rho = n = |\Psi|^2$$

GP: $F(\rho) \equiv \lambda\rho$

Log: $F(\varrho) \equiv \beta^{-1}\ln(\Omega\varrho)$

Field-theoretical action (without the chemical-potential term " $-\mu|\Psi|^2$ "):

GP: $V(\Psi) = \dfrac{\lambda}{2}|\Psi|^4$

Log:

$$V(\Psi) = \frac{\beta^{-1}}{\Omega}\left\{\Omega|\Psi|^2\left[\ln\left(\Omega|\Psi|^2\right) - 1\right] + 1/2\right\}$$

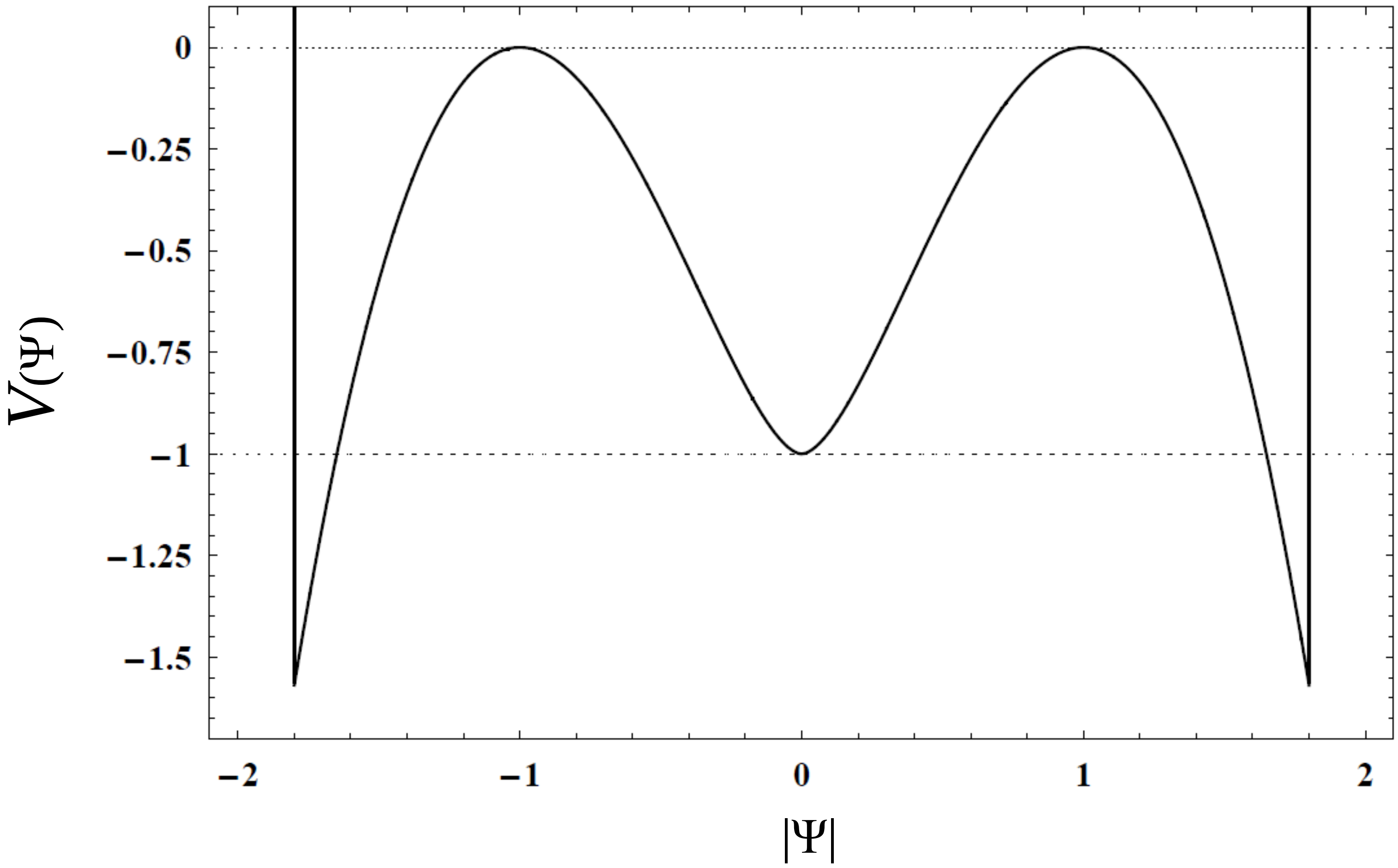
$$= \tfrac{1}{2}\beta^{-1}\Omega|\Psi|^4 - \beta^{-1}|\Psi|^2 + \mathcal{O}\left[(\Omega|\Psi|^2 - 1)^3\right]$$

GP is one of perturbative limits of Log

$$V(\Psi) = \frac{\beta^{-1}}{\Omega}\left\{\Omega|\Psi|^2\left[\ln\left(\Omega|\Psi|^2\right) - 1\right] + 1/2\right\}$$

LogSE's Field potential (cont'd):

Normalization condition => "Walls"



- Log potential is smooth but non-Taylor-expandable for small densities (|Ψ| -> 0)
- GP is one of the perturbative limits of Log (expand near non-zero extrema)

“Hydrodynamic” E.O.S. (leading-order w.r.t. $\hbar$): $\rho = n = |\Psi|^2$

GP: $$p - p_0 = \frac{\lambda}{2} n^2$$

Log: $$p - p_0 = -(m\beta)^{-1} n$$

Ideal-gas law
(Clapeyron-Mendeleev)

General quantum fluid: $$p = p_0 + k_1 n + k_2 n^2 + \ldots$$

…can be used to derive the Log fluid

Speed of <u>sound</u> (leading-order w.r.t. $\hbar$):

GP: $$c_s = \sqrt{\lambda}\sqrt{n}$$

Log: $$c_s = 1/\sqrt{m|\beta|}$$

doesn’t depend on density (this finds use in post-relativistic theories of *superfluid vacuum*)

What we have learnt:

- ❑ Perturbatively equivalent BEC models, GP and Log, differ drastically when treated in a non-perturbative way
- ❑ Log BEC is more “ideal” than GP
- ❑ Speed of wave-like fluctuations is constant for Log BEC

$$\frac{\hbar^2}{2m}\Delta\Psi + \left[\mu + \beta^{-1}\ln\left(a^3\Psi^2\right)\right]\Psi = 0$$

Ground state

We already know that the ground-state solution of GP-BEC in empty space (no external potential) implies that the particle density is constant i.e. GP condensate is uniform and tries to occupy all available volume.

What about Logarithmic Bose liquid? Earlier we have found:

$$n(r) = \Psi^2 = \frac{N}{\pi^{3/2}a_\beta^3}e^{-(r/a_\beta)^2}$$

self-sustainable,
localized,
stable

$$\mu = 3\beta^{-1}\left[1 - \ln\left(\frac{N^{1/3}}{\sqrt{\pi}}\frac{a}{a_\beta}\right)\right] = -\beta^{-1}\ln\left(N/N_0\right)$$

where:

$$N_0 \equiv (e\sqrt{\pi}a_\beta/a)^3$$

$$a_\beta = \sqrt{A_\beta} = \hbar\sqrt{\beta/2m}$$

Observation: LogBEC behaves more like liquid than gas (good news)

III. Applications: Superfluid ^4He

Introduction. Liquid Helium

- Helium-4 becomes liquid below 4.22 K (the *He I phase*) at 1 atm.
- Below 2.17 K it becomes superfluid (the *He II phase*): zero viscosity, very high thermal conductivity. Two-fluid approach: He II is a mixture of atoms in a ground state (*superfluid component*) and atoms in excited states (*ordinary liquid component*).
- Helium remains liquid down to absolute zero at normal pressures (zero-point energy of the system is too high to allow solidification).

Surprisingly, NO theory of the superfluid component of ^4He exists as yet (complete, commonly accepted)

Instead:

Eclectic approach:

There exists a number of different theories and approaches which:

1. Explain only certain sides of the phenomenon

2. Relay on different assumptions

3. Conflict with each other

Classification

Empirical or semi-empirical theoretical approaches:

Pros: fit experimental data well

Contras: using significant bulk of experimental information (e.g., empirical intermolecular potentials) => not actually a complete theory (an experiment is being explained by another experiment, with additional assumptions)

Examples: *Apaja & Saarela'1998* (using the *Aziz* potential, etc)

Fundamental theoretical approaches:

Pros: Clear understanding of underlying physics, minimum of assumptions

Contras: do not fit all experimental data (often only qualitatively)

Examples:

- vortex rings [*Feynman'54, Feynman&Cohen'56*]
- hard spheres [*Lee,Huang&Yang'57, Liu&Wong'64,Ivashin&Poluektov'11*]
- stochastic clusters [*Kruglov&Collett'01*]
- lattice liquid [*Nozieres'04*]

Requirements for the theory

1. Must consider the non-locality of the superfluid.
   Indeed, at low T's the thermal de Broglie wavelengths of atoms become larger than inter-atomic separation => wf's overlap => each of the "atoms" is not an isolated system anymore.
   Therefore: (**1**) open quantum system formalism must be engaged (although, the environment is peculiar – not thermal bath, not noise), (**2**) new (collective) degrees of freedom are needed.

2. Must describe the atomic-scale continuity of the superfluid.
   (unlike classical fluids, the quantum ones show no discreteness at much shorter scales) [Adamenko,Nemchenko,Tanatarov'03] => physical definition of a notion of the fluid element/parcel is needed.

3. Must take into account that superfluid He contains BEC of helium atoms (but not 100%)

Our Assumption

Strongly interacting helium atoms at low T form the bound states described by LogSE (note: NO external/trapping potential)

$$\left[-i\hbar\,\partial_t - \frac{\hbar^2}{2m}\vec{\nabla}^2 - \beta^{-1}\ln\left(\tilde{a}^3|\Psi|^2\right)\right]\Psi = 0,$$

where $\Psi = \Psi(\vec{x}, t)$ is the wavefunction of condensate normalized to the number of particles $\widetilde{N}$ - such that particle density is determined as $n = |\Psi|^2$, m is the mass of the constituent particle (helium atom in our case, i.e., $m = m_{\mathrm{He}^4} \approx 6.64 \times 10^{-24}$ g), and β and $\tilde{a}$ are constant

Immediate consequence

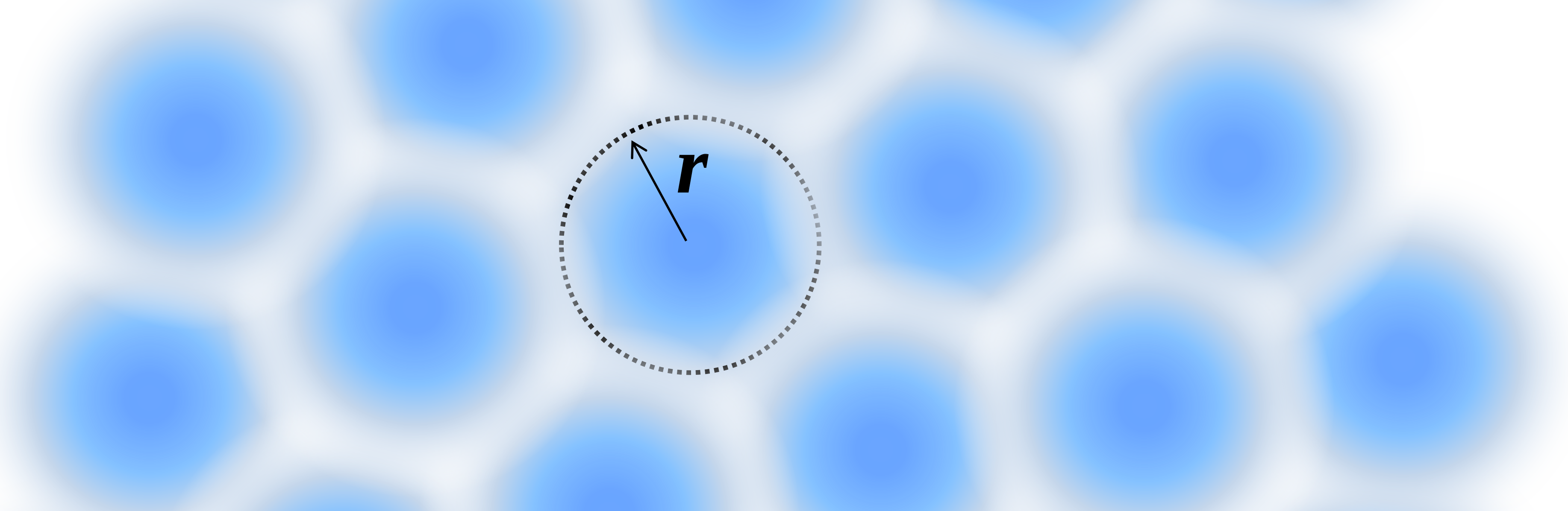
Ground state of LogSE = Gaussian droplet (no classical surface) = Fluid element/parcel (our new D.O.F.)

density: $$n(\vec{x}) = n(0)\,\mathrm{e}^{-(r/a)^2},$$

$n(0) = \widetilde{N}/(\pi^{3/2}a^3)$ $a = \hbar\sqrt{\beta/2m}$ is the characteristic radius

Physical picture (qualitative)

We see the superfluid He-4 not as a homogeneous logarithmic Bose liquid but as the “gas” of the fluid elements whose “interiors” are described by the logarithmic liquid. …Clusterization (sort of).



Each fluid element is a **Gaussian** of size $r \sim a$ (self-sustained, stable, no border, no surface tension).

The “interior” of each such element is described by LogSE (shorter-wavelength part) but the inter-element interaction requires a separate derivation (longer-wavelength model).

Long-wavelength part

Each fluid element is Gaussian cluster (not point-like!) but we use the long-wavelength approximation: we encode the elements' extendedness into the inter-particle potential and deal with them like with point-like objects. Thus, altogether we get a two-scale theory (cf. "*Russian-doll*" nesting of scales).

$$\hat{H} = \int d^3x\, \hat{\psi}^\dagger(\vec{x}) \left(-\frac{\hbar^2}{2M} \vec{\nabla}^2 \right) \hat{\psi}(\vec{x}) + \hat{H}_{\rm int},$$

where M is the mass of the volume element and $\hat{\psi}$ is the corresponding field operator. The interaction is defined via the nonlocal term

$$\hat{H}_{\rm int} = \frac{1}{2} \int\int d^3x\, d^3x'\, \hat{\psi}^\dagger(\vec{x}) \hat{\psi}^\dagger(\vec{x}') U(|\vec{x} - \vec{x}'|)\, \hat{\psi}(\vec{x}') \hat{\psi}(\vec{x}),$$

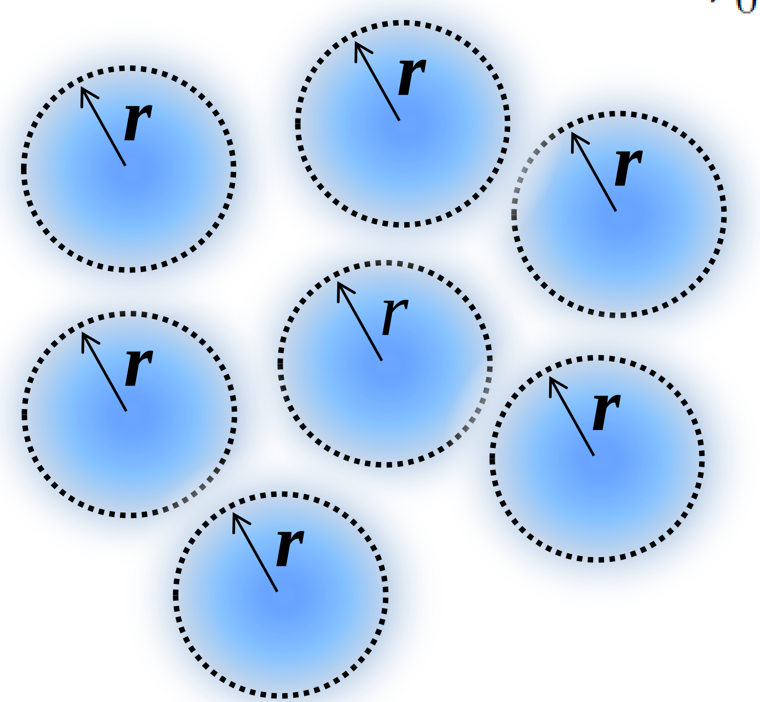
where $U(r)$ is the energy of interaction between volume elements

Deriving inter-element interaction potential

Gaussian volume element of size $r \sim a$ stores an amount of internal bulk mass-energy

$$\epsilon(r) \propto \int_0^r n(r')r'^2 dr' \propto \frac{1}{a}(r - r_0)\, \mathrm{e}^{-(r/a)^2} \left[1 + \mathcal{O}\left(r - a\right)\right]$$

$$r_0 = a\left[1/2 + 1/(\mathrm{e}\sqrt{\pi}\,\mathrm{erf}(1))\right] \approx 0.75\, a$$



Each element is stable => tries to maintain its size

Change of size = change of the energy $\epsilon(r)$

System is isolated (thanx, Dewar) hence the energy can come from neighbors only

$$U(|\vec{x} - \vec{x}'|) \propto \epsilon(|\vec{x} - \vec{x}'|)$$

Thus, we can introduce the proportionality constant and finally obtain that the inter-element interaction potential takes the form:

$$U(r) = \frac{U_0}{a}(r - r_0)\, e^{-(r/a)^2}$$

where

$$r_0 = a\,[1/2 + 1/(e\sqrt{\pi}\,\mathrm{erf}(1))] \approx 0.75\,a$$

quantity $U_0 = -aU(0)/r_0 \approx -1.34\,U(0)$ becomes the free parameter of the long-wavelength part of the theory. If U_0 is positive then the critical radius r_0 determines the inter-element separation below which a pair of neighboring volume elements becomes unstable against coalescence.

Observables:
Theory vs. Experiment

Energy spectrum

To derive the energy of excitations one can use three methods:

- Bogolyubov transform [any textbook on BEC]
- perturbation theory [Brueckner&Sawada'57]
- linear fluctuations of wave equation [Ivashin&Poluektov'11]

Leading order: same expression for the energy of a quasi-particle.

$$\hat{H} \approx \frac{1}{2} n_0^2 \bar{U}_0 V + \sum_{\vec{p}\neq 0} E_p \hat{a}_p^\dagger \hat{a}_p$$

where

$$E_p = \frac{p^2}{2M}\sqrt{1+\frac{4n_0 M \bar{U}_p}{p^2}}$$

where $\hat{a}_p^\dagger$ and $\hat{a}_p$ are the creation and annihilation operators of the quasi-particle with momentum $\vec{p}$, $n_0 = N/V = \rho_{\mathrm{He}^4}/M$ is the background particle density of the liquid of N elements occupying the volume V

our case: $$\bar{U}_p = \int U(x)\, \mathrm{e}^{i\vec{p}\cdot\vec{x}/\hbar} d^3x = \pi a^3 U_0 \left(1 - \sqrt{\pi} f_k\, \mathrm{e}^{-(ak/2)^2}\right)$$

$$f_k = r_0/a + \left[\tfrac{1}{2}ak - (ak)^{-1}\right] \mathrm{erfi}(ak/2) \qquad \vec{k} = \vec{p}/\hbar$$

$$E_p = \frac{p^2}{2M}\sqrt{1+\frac{4n_0 M\bar{U}_p}{p^2}}$$

$$\bar{U}_p = \pi a^3 U_0 \left(1-\sqrt{\pi} f_k \, e^{-(ak/2)^2}\right)$$

Energy spectrum (cont'd)

Write spectrum in dimensionless form: using the de Broglie scale of the fluid element

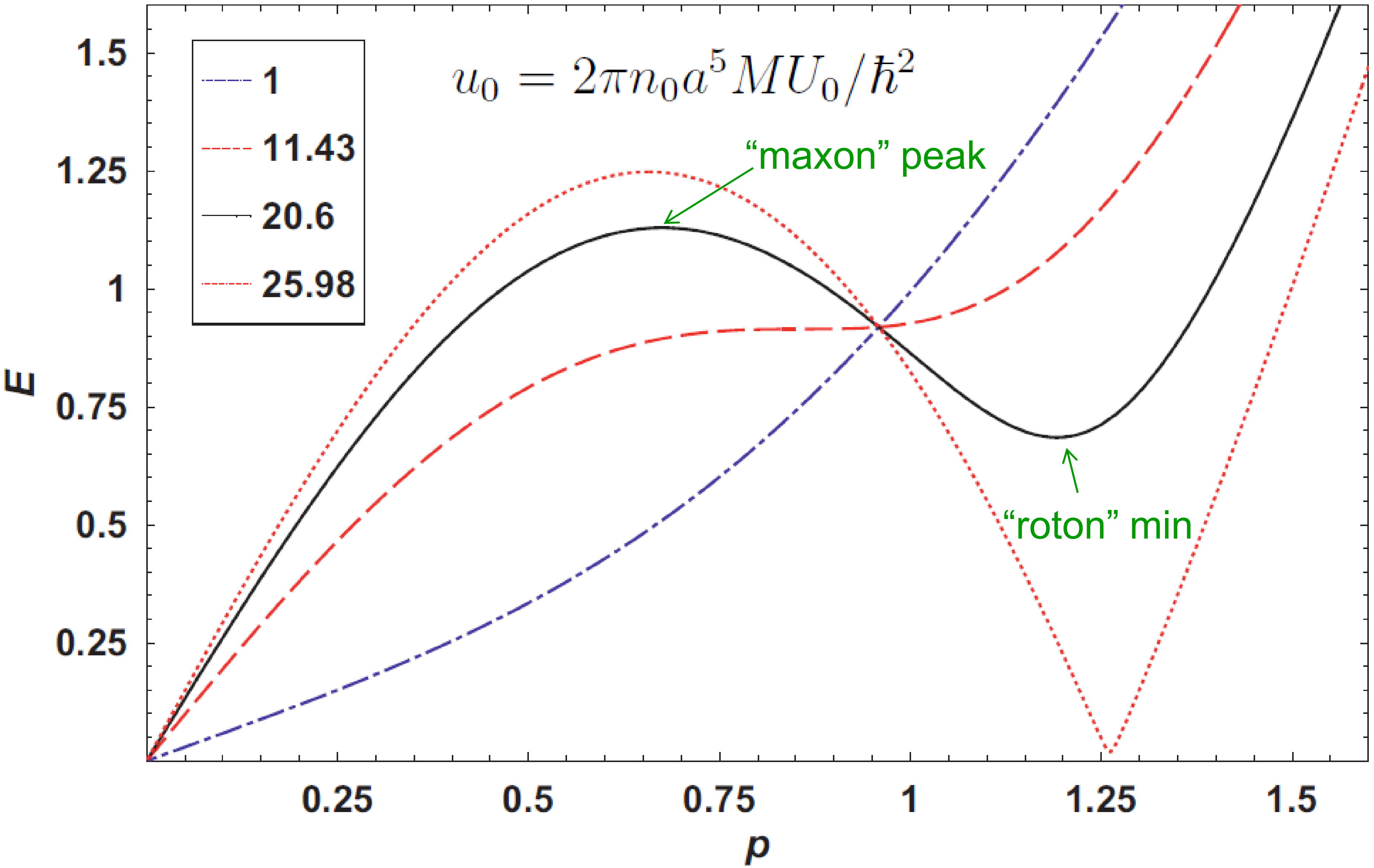
$$p_a = 2\hbar/a \qquad E_a = p_a^2/2M$$

Turns out that the behavior of the d/less energy excitations depends on only one parameter:

$$u_0 = 2\pi n_0 a^5 M U_0/\hbar^2$$

$$E_p = \frac{p^2}{2M}\sqrt{1+\frac{4n_0 M\bar{U}_p}{p^2}}$$

$$\bar{U}_p = \pi a^3 U_0 \left(1 - \sqrt{\pi} f_k \, \mathrm{e}^{-(ak/2)^2}\right)$$

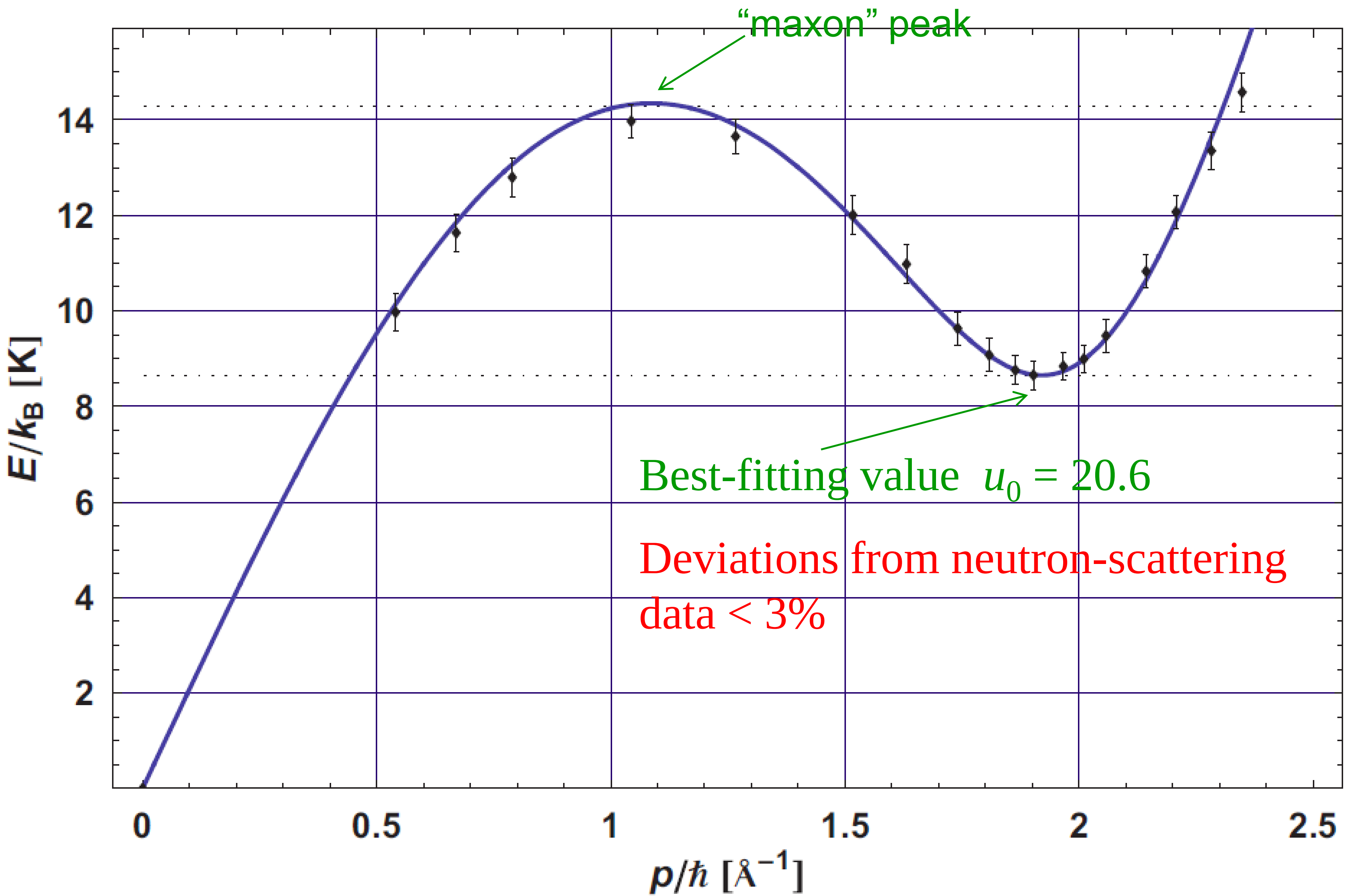


Landau "roton" spectrum: between 11.43 and 25.98 Notice: no rotational d.o.f. used

$$E_p = \frac{p^2}{2M}\sqrt{1+\frac{4n_0 M\bar{U}_p}{p^2}}$$

$$\bar{U}_p = \pi a^3 U_0\left(1-\sqrt{\pi} f_k\, e^{-(ak/2)^2}\right)$$

$$u_0 = 2\pi n_0 a^5 M U_0/\hbar^2$$



$$E_p = \frac{p^2}{2M}\sqrt{1 + \frac{4n_0 M \bar{U}_p}{p^2}}$$

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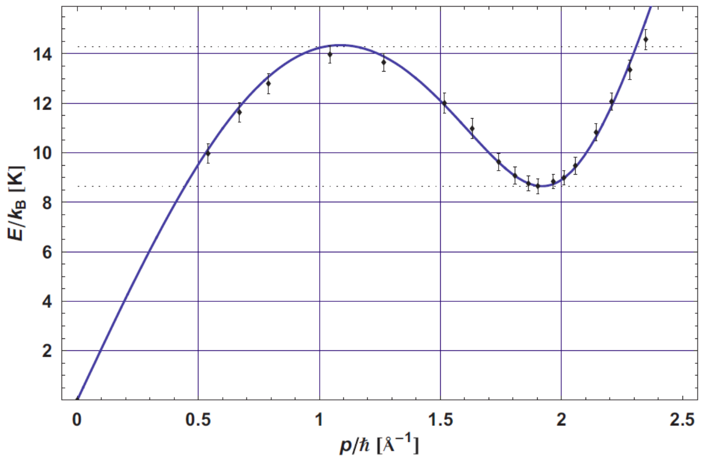


Once we have determined the right value of u_0 we can numerically deduce all other parameters from the corresponding spectral curve:

$$\bar{a} \approx 2.37 \text{ Å}, \; M/m \approx 1.22, \; U_0/\Delta \approx 69,$$
$$a \approx 1.25 \text{ Å}, \; p_a/p_0 \approx 0.84, \; E_a/\Delta \approx 1.46, \qquad (*)$$
$$\beta^{-1}/k_B \approx 3.84 \text{ K}, \; \beta^{-1}/E_a \approx 0.31,$$

The set (*) is unique, it will be used for further comparison with experiments.

$\bar{U}_p = \pi a^3 U_0 \left(1 - \sqrt{\pi} f_k \, \mathrm{e}^{-(ak/2)^2}\right)$ $f_k = r_0/a + \left[\tfrac{1}{2}ak - (ak)^{-1}\right] \mathrm{erfi}(ak/2)$

Structure factor

If one naively applies the standard Feynman formula, one obtains:

$$S_k^{(0)} = p^2/2ME_p = \left(1 + \frac{4n_0 M \bar{U}_p}{p^2}\right)^{-1/2}$$

However, this formula's derivation procedure implies that d.o.f. are He atoms (in our model it should be the Gaussian elements), and also it doesn't take into account the log nonlinearity. Thus, formula must be modified. It's easy: take the original derivation but replace atoms with elements and include the log-induced effects.

We begin with the total energy of the system of *N* fluid elements which comes from minimizing the integral:

$$\mathcal{E} = \int \psi^* \mathcal{H} \psi \, d^{3N}x \quad \text{at fixed} \quad \mathcal{J} = \int \psi^* \psi \, d^{3N}x$$

where the *N*-body Hamiltonian: $\mathcal{H} = -(\hbar^2/2M) \sum_i \vec{\nabla}_i^2 + V,$

$\bar{U}_p = \pi a^3 U_0 \left(1 - \sqrt{\pi} f_k \, \mathrm{e}^{-(ak/2)^2}\right)$ $\quad f_k = r_0/a + \left[\frac{1}{2}ak - (ak)^{-1}\right] \mathrm{erfi}(ak/2)$

For the excited-state wavefunction one uses the Bijl ansatz:

$$\psi = F\psi_0 = \sum_i f(\vec{x}_i)\psi_0, \quad f(\vec{x}) \equiv \mathrm{e}^{i\vec{k}\cdot\vec{x}}$$

The only difference: $\psi_0 = \psi_0(\vec{x}^N)$ is now the ground-state wavefunction of the system - must satisfy the stationary N-body LogSE:

$$\mathcal{H}_0\psi_0 = \frac{m}{M}\left[NE_0 + \beta^{-1}\ln\left(\tilde{a}^{3N}|\psi_0|^2\right)\right]\psi_0$$

where the potential part has been replaced by the logarithmic term, according to the two-scale structure of our theory (the trapping potential is neglected)

$$E_0 = 3\beta^{-1}\left[1 + \ln\left(\sqrt{\pi}a/\tilde{a}\right)\right] \approx 3\beta^{-1}\left(1 + \tfrac{1}{2}\ln\pi\right)$$

Exact ground-state solution is a product of 1-particle wavefunctions:

$$\psi_0 \propto \prod_i \Psi(\vec{x}_i) \propto \prod_i \exp\left(-|\vec{x}_i|^2/2a^2\right)$$

$\boxed{E_0 = 3\beta^{-1}\left[1 + \ln\left(\sqrt{\pi}a/\tilde{a}\right)\right]}$ $\boxed{S_k \equiv \int P(\vec{x})\mathrm{e}^{i\vec{k}\cdot\vec{x}}d^3x}$

Performing the computation *a la* Feynman…

$$E_p + E_0 = \frac{\int \left[\frac{\hbar^2}{2M}\frac{1}{N}\sum_i \vec{\nabla}_i F^* \cdot \vec{\nabla}_i F + \frac{m}{M}|F|^2\left(E_0 + \beta^{-1}\ln\left(\tilde{a}^3\rho_N^{1/N}\right)\right)\right]\rho_N d^{3N}x}{\int |F|^2 \rho_N d^{3N}x}$$

$$\rho_N = \left(\frac{1}{\pi^{3/2}a^3}\right)^N \mathrm{e}^{-\sum_i |\vec{x}_i|^2/a^2}$$

$$E_p + E_0 - \frac{m}{M}\left[E_0 + \beta^{-1}\ln\left(\tilde{a}^3\rho_N^{1/N}(0)\right)\right] = \frac{\hbar^2}{2M}\frac{\frac{1}{N}\sum_i \int \vec{\nabla}_i F^* \cdot \vec{\nabla}_i F \rho_N d^{3N}x}{\int |F|^2 \rho_N d^{3N}x}$$

$$E_p + \Sigma = \frac{\hbar^2}{2M}\frac{\int \vec{\nabla} f^*(\vec{x}) \cdot \vec{\nabla} f(\vec{x}) d^3x}{\int f^*(\vec{x}_1) f(\vec{x}_2) P(\vec{x}_1 - \vec{x}_2) d^3x_1 d^3x_2}$$

$$\Sigma \approx E_0 - \frac{m}{\beta M}(3 + \ln\pi) \approx 3\beta^{-1}\left[1 - \frac{m}{M} + \frac{1}{2}\left(1 - \frac{2}{3}\frac{m}{M}\right)\ln\pi\right]$$

$$(E_p + \Sigma)\int f(\vec{x}_2) P(\vec{x}_1 - \vec{x}_2) d^3x_2 + \frac{\hbar^2}{2M}\vec{\nabla}^2 f(\vec{x}_1) = 0$$

$E_0 = 3\beta^{-1}\left[1+\ln\left(\sqrt{\pi}a/\tilde{a}\right)\right]$ $\vec{k} = \vec{p}/\hbar$ $S_k \equiv \int P(\vec{x}) e^{i\vec{k}\cdot\vec{x}} d^3x$

Performing the computation *a la* Feynman, we eventually obtain:

$$S_k = \frac{p^2}{2M\left(E_p+\Sigma\right)} = \left[\sqrt{1+\frac{4n_0 M\bar{U}_p}{p^2}} + \frac{2M\Sigma}{p^2}\right]^{-1}$$

additional contribution due to the Log nonlinearity

where:

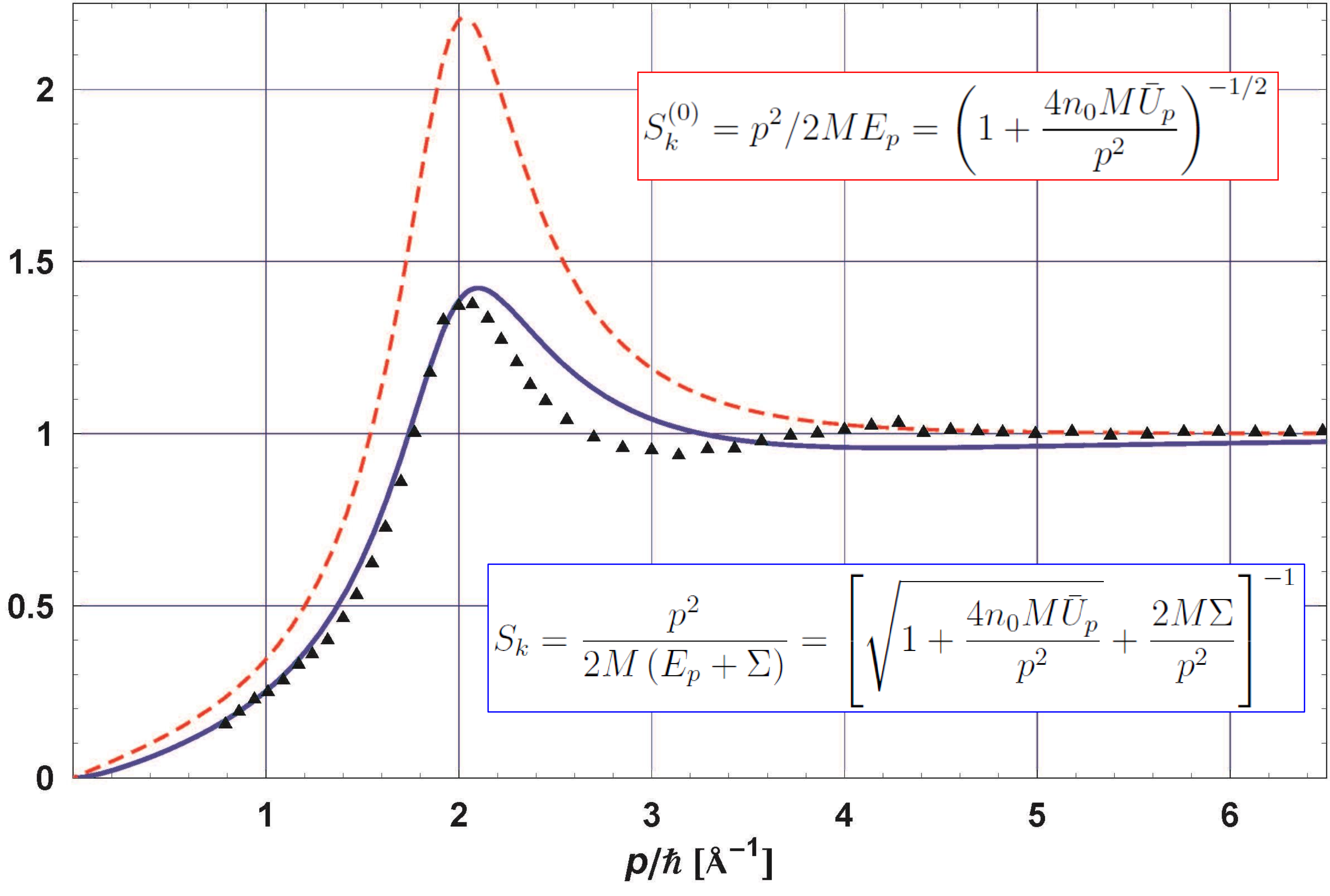
$$\Sigma \approx E_0 - \frac{m}{\beta M}\left(3+\ln\pi\right) \approx 3\beta^{-1}\left[1-\frac{m}{M}+\frac{1}{2}\left(1-\frac{2}{3}\frac{m}{M}\right)\ln\pi\right]$$

$$E_p = \frac{p^2}{2M}\sqrt{1+\frac{4n_0 M\bar{U}_p}{p^2}} \qquad n_0 = N/V = \rho_{\mathrm{He}^4}/M$$

$$\bar{U}_p = \int U(x)\, e^{i\vec{p}\cdot\vec{x}/\hbar} d^3x = \pi a^3 U_0\left(1-\sqrt{\pi}f_k\, e^{-(ak/2)^2}\right)$$

$$f_k = r_0/a + \left[\tfrac{1}{2}ak - (ak)^{-1}\right]\mathrm{erfi}(ak/2)$$

Structure factor: Theory vs experiment [*Svensson'89*]

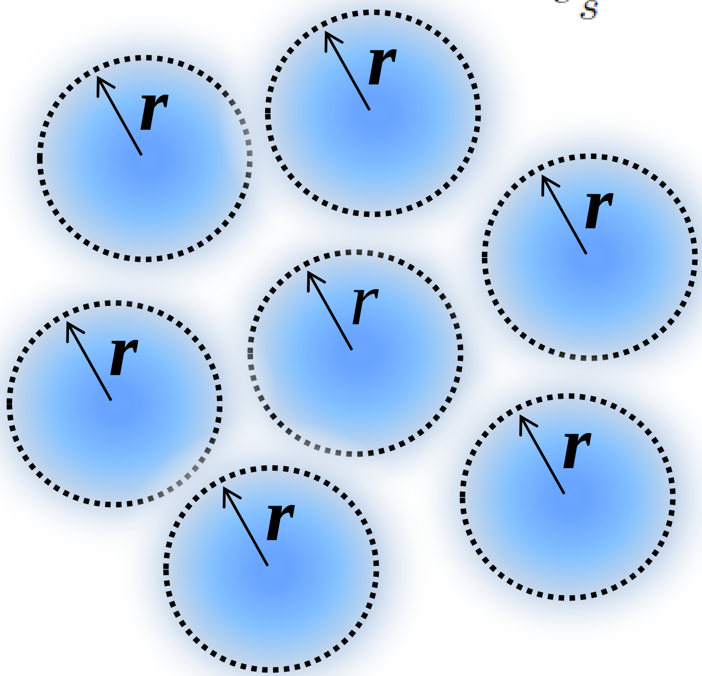


$\bar{U}_p = \pi a^3 U_0 \left(1 - \sqrt{\pi} f_k \, \mathrm{e}^{-(ak/2)^2}\right)$ $\quad$ $f_k = r_0/a + \left[\frac{1}{2}ak - (ak)^{-1}\right] \mathrm{erfi}(ak/2)$

Speed of sound

If elements were point-like and structureless – apply the textbook BEC formula:

$$v_s^{(0)} = \sqrt{n_0 \bar{U}_0 / M} \qquad v_s^{(0)} \approx 273 \text{ m/s}$$



But they are not => two contributions:

excitation: $v_s^{(0)} = \sqrt{n_0 \bar{U}_0 / M}$

background: $c_s = 1/\sqrt{m|\beta|}$ $\quad c_s \approx 89 \text{ m/s}$

Estimate: take these with Gaussian weights (density) and do average.

Weights: $\chi = (1/2)\, n(\vec{x})/n(0) = \frac{1}{2}\mathrm{e}^{-(r/a)^2}$

$$\langle v_s \rangle = \bar{a}^{-1} \int_0^{\bar{a}} \left[\chi c_s + (1-\chi)\, v_s^{(0)}\right] dr = v_s^{(0)} \left[1 - \frac{\sqrt{\pi}}{4} \frac{a}{\bar{a}} \mathrm{erf}(\bar{a}/a) \left(1 - \frac{c_s}{v_s^{(0)}}\right)\right]$$

Speed of sound (cont'd)

$$\langle v_s \rangle = \bar{a}^{-1} \int_0^{\bar{a}} \left[\tfrac{1}{2}\chi c_s + \left(1 - \tfrac{1}{2}\chi\right) v_s^{(0)}\right] dr = v_s^{(0)} \left[1 - \frac{\sqrt{\pi}}{4}\frac{a}{\bar{a}}\,\mathrm{erf}(\bar{a}/a)\left(1 - \frac{c_s}{v_s^{(0)}}\right)\right]$$

$$v_s^{(0)} = \sqrt{n_0 \bar{U}_0 / M} \approx 273 \ \mathrm{m/s}$$

$$c_s = 1/\sqrt{m|\beta|} \approx 89 \ \mathrm{m/s}$$

$$\chi = n(\vec{x})/n(0) = \mathrm{e}^{-(r/a)^2}$$

$$\tfrac{4}{3}\pi\bar{a}^3 = 1/n_0 = M/\rho_{\mathrm{He}^4}$$

Finally, theoretical value :

$$\boxed{\langle v_s \rangle \approx 231 \ \mathrm{m/s}}$$

(cf. experiment: 237±2 => deviation < 3%)

IV. General conclusions
and
Future directions

- Here we proposed a theory of structure and excitations in superfluid helium which consists of two models which act on different length scales, but are connected via the parametric space: the behaviour of quantities and the values of parameters in the long-wavelength model are derived from the short-wavelength part (cf. “Russian-doll” nesting of scales).

- The short-wavelength part advocates the appearance of the collective d.o.f.’s which justify the possibility of the fluid-dynamical description and can be used to characterize the volume elements of the quantum liquid.

- The “interior” structure of a fluid volume element is described in short-wavelength part by means of the non-perturbative approach based on the logarithmic wave equation. It turns out that the interior density of the element obeys the Gaussian distribution.

- The long-wavelength part is the quantum many-body theory of the volume elements as effectively point-like objects - yet their spatial extent and internal structure are taken into account by virtue of the nonlocal interaction term. The corresponding inter-particle interaction potential was not postulated but derived from the short-wavelength part.

➢ In our theory the quantitative agreement with various experimental data has been achieved: with only one essential parameter in hand we reproduced at high accuracy (better than three per cent) not only the roton minimum but also the neighboring local (maxon) maximum, as well as the velocity of sound and structure factor.

Future research directions:

➢ Regarding He: temperature effects (heat transfer, 2nd sound) to be studied.

➢ In general: it would be interesting to study the applications of the models with the non-polynomial wave equations and Gaussian-like inter-particle potentials in the physical situations when superfluid becomes effectively low-dimensional (cigar- or disk-shaped) and/or subjected to external fields or admixtures.

➢ Mathphysics problem: is there any way to derive LogSE from the N-body Hamiltonian with some peculiar interparticle pot-al (by analogy with GP $\leftrightarrow$ delta-like)?

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THANK YOU !