Solvable Models in Ultracold Physics III

Itzhak Roditi

Centro Brasileiro de Pesquisas Físicas
Rio de Janeiro, Brasil

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Monodromy matrix $T(u) \to N \times N$ matrix taking values in an algebra $\mathcal{A}$:

$$T(u) = \sum_{i,j=1}^{N} T_{ij}(u)\, E_{ij} \tag{1}$$

$$T_{ij}(u) \in \mathcal{A}\,;$$

$$E_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1_{\{ij\}} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{2}$$

Here we follow closely Foerster and Ragoucy NPB 2007.

$T(u)$ satisfies:

$$R_{12}(u-v)\, T_1(u)\, T_2(v) = T_2(v)\, T_1(u)\, R_{12}(u-v), \tag{3}$$

where $T_1(u) = T(u) \otimes \mathbb{I}_N$.

The $R-$matrix obeys the Yang-Baxter equation:

$$\begin{aligned} R_{12}(u_1-u_2)\, R_{13}(u_1-u_3)\, R_{23}(u_2-u_3) \\ = \\ R_{23}(u_2-u_3)\, R_{13}(u_1-u_3)\, R_{12}(u_1-u_2)\,. \end{aligned} \tag{4}$$

It is possible to produce a T for several sites

$$T(u) = T^{(1)}(u)\, T^{(2)}(u) \cdots\, T^{(M)}(u),$$

where the superscript indicates in which copy of the algebra the monodromy matrix acts.

Spin chain models, for instance, may be seen in this way, as the copies of the algebra defining the M quantum spaces. The $T^{(j)}(u)$ are called an elementary monodromy matrix, the product of all such matrices provides the 'full' monodromy matrix.

It is important to notice that in a physical model, one has to represent the monodromy matrix, which means to choose for the $T^{(j)}(u)$ a representation of the algebra $\mathcal{A}$: different representations will be related to different physical models. From the RTT relation, for $t(u) = Tr(T(u))$:

$$[t(u), t(v)] = 0$$

Hermiticity of T

We will be studying hermitian elementary monodromy matrices.

$$(T(u))^{\dagger} = T(u) \tag{5}$$

$$T^{\dagger}_{ij}(u) = T_{ji}(u)$$

Which implies

$$(T(u))^{\dagger} = \left(T^{(1)}(u)\,T^{(2)}(u)\right)^{\dagger} = T^{(2)}(u)\,T^{(1)}(u) \tag{6}$$

The full monodromy matrix is then not hermitian.
Nevertheless:

$$t^{\dagger}(u) = tr\left(T^{(2)}(u)\,T^{(1)}(u)\right) = tr\left(T^{(1)}(u)\,T^{(2)}(u)\right) = t(u) \tag{7}$$

i.e., the transfer matrix is hermitian. When $M = 2$(or 1) this property is always valid. For BEC related models, one usually studies only the cases $M = 2$.

$gl(N)$ models

R-matrix is written as:

$$R_{12}(x) = \mathbb{I} \otimes \mathbb{I} - \frac{1}{x} P_{12}, \tag{8}$$

where

$$P_{12} = \sum_{i,j=1}^{N} E_{ij} \otimes E_{ji}$$

is the permutation operator.

Spin chains

One takes the elementary monodromy matrices as

$$L(u) = \sum_{i,j=1}^{N} L_{ij}(u)\, E_{ij} \;\rightarrow\; L_{ij}(u) = u\delta_{ij} + e_{ij} \tag{9}$$

or

$$L(u) = \begin{pmatrix} u + e_{11} & e_{12} & e_{13} & \dots & e_{1N} \\ e_{21} & u + e_{22} & e_{23} & \dots & e_{2N} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ e_{N-2,1} & & & & e_{N-1,N} \\ e_{N1} & e_{N2} & \dots & e_{N,N-1} & u + e_{NN} \end{pmatrix} \tag{10}$$

Here, e_{ij} are $gl(N)$ abstract generators obeying

$$[e_{ij}\, ,\, e_{kl}] = \delta_{jk}\, e_{il} - \delta_{il}\, e_{kj} \tag{11}$$

$L(u)$ obeys the RTT relation (3) with the R-matrix (8).
$e_{ij}^{\dagger} = e_{ji}$ ensures the Hermiticity of $L(u)$. It is possible to use an automorphism $u \to u + w$ that has the effect of providing extra free parameters.
Therefore, one has

$$T(u) = L^{(1)}(u + w_1)\, L^{(2)}(u + w_2)$$

The hermitian transfer matrix is:

$$t(u) = \sum_{j=1}^{N} (u + w_1 + e_{jj}^{[1]}) \otimes (u + w_2 + e_{jj}^{(2)}) + \sum_{j \neq k}^{N} e_{jk}^{[1]} \otimes e_{kj}^{(2)}\, . \tag{12}$$

The choice of a $gl(N)$ representation for each of the sites will dictate the physical model.
For highest weight finite dimensional reps, it leads to spin chains models.
If one chosses the fundamental representation of $gl(N)$

$$\pi(e_{ij}) = E_{ij}\,,\ i,j = 1,\dots,N \tag{13}$$

for both elementary monodromy matrices, it leads to a two-site spin chain. For the particular case of $gl(2)$, one recovers the Pauli matrices

$$\pi(e_{12}) = \sigma_+\ ;\ \pi(e_{21}) = \sigma_-\ ;\ \pi(e_{11} - e_{22}) = \sigma_z\ ;\ \pi(e_{11} + e_{22}) = \mathbb{I}_2 \tag{14}$$

and

$$H = t(0) = \sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+ + \frac{1}{2}\sigma_z \otimes \sigma_z + \frac{1}{2} \tag{15}$$

Spin 1

Another representation (spin 1)

$$\pi(e_{12}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = S_+ \ ; \ \pi(e_{21}) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = S_- \ ; \tag{16}$$

$$S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \ ; \ \pi(e_{11}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}(S_z + \mathbb{I}_3) \ ; \tag{17}$$

$$\pi(e_{22}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}(\mathbb{I}_3 - S_z)$$

The Hamiltonian is:

$$H = t(0) = S_+ \otimes S_- + S_- \otimes S_+ + \frac{1}{2} S_z \otimes S_z + \frac{1}{2}$$

Oscillatory reps of $gl(N)$

creation/annihilation operators $(a_i, a_i^\dagger)$,

$$[a_i, a_j^\dagger] = \mu_i\,\delta_{ij} \quad ; \quad [a_i^\dagger, a_j^\dagger] = [a_i, a_j] = 0\,. \tag{18}$$

The following, $\mathcal{L}(u)$, given by

$$\mathcal{L}(u) = \sum_{i,j=1}^{N} \mathcal{L}_{ij}(u)\, E_{ij} \text{ with } \mathcal{L}_{ij}(u) = \mu_i\, u\, \delta_{ij} + \frac{q_i}{q_j}\, a_i^\dagger\, a_j \tag{19}$$

satisfies

$$[\mathcal{L}_{ij}(u)\,,\, \mathcal{L}_{kl}(v)] = \frac{1}{u-v}\left(\mathcal{L}_{kj}(u)\mathcal{L}_{il}(v) - \mathcal{L}_{kj}(v)\mathcal{L}_{il}(u)\right). \tag{20}$$

Which is the same as the RTT relation,

$$R_{12}(u-v)\,\mathcal{L}_1(u)\,\mathcal{L}_2(v) = \mathcal{L}_2(v)\,\mathcal{L}_1(u)\,R_{12}(u-v) \qquad (21)$$

where,

$$R_{12}(x) = \mathbb{I}\otimes\mathbb{I} - \frac{1}{x}\,P_{12} \qquad (22)$$

so that

$$T(u) = \mathcal{L}^{(1)}(u+w_1)\mathcal{L}^{(2)}(u+w_2)$$

gives a solvable model.

It corresponds to a choice of a $gl(N)$ representation

$$\pi(e_{ij}) = a_i^\dagger \, a_j \qquad i, j = 1, ..., N \tag{23}$$

for the elementary monodromy matrices Eq.9. The highest weight being the Fock space vacuum $|0>$.
For hermitian elementary matrices, the expression for $\mathcal{L}$ is

$$\mathcal{L}_{ij}(u) = \mu_i \, u \, \delta_{ij} + a_i^\dagger \, a_j \text{ with } \mu_j \in \mathbb{R} \tag{24}$$

$$\mathcal{L}(u) = \begin{pmatrix} \mu_1 \, u + n_1 & a_1^\dagger \, a_2 & a_1^\dagger \, a_3 & \dots & a_1^\dagger \, a_N \\ a_2^\dagger \, a_1 & \mu_2 \, u + n_2 & a_2^\dagger \, a_3 & \dots & a_2^\dagger \, a_N \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{N-2}^\dagger \, a_1 & & & & a_{N-1}^\dagger \, a_N \\ a_N^\dagger \, a_1 & a_N^\dagger \, a_2 & \dots & a_N^\dagger \, a_{N-1} & \mu_N \, u + n_N \end{pmatrix} \tag{25}$$

$$n_i = a_i^\dagger \, a_i$$

Transfer matrices

The matrices $L(u)$ and $\mathcal{L}(u)$, lead to different types of transfer matrices:

$$t(u) = tr\left(L^{(1)}(u+w_1)\,L^{(2)}(u+w_2)\right)$$
$$\sim u\sum_{i=1}^{N}\left(E_{ii}^{(1)}+E_{ii}^{(2)}\right)+\sum_{i,j=1}^{N}E_{ij}^{(1)}\,E_{ji}^{(2)}+\sum_{i=1}^{N}\left(w_2\,E_{ii}^{(1)}+w_1\,E_{ii}^{(2)}\right) \tag{26}$$

$$t(u) = tr\left(L^{(1)}(u+w_1)\,\mathcal{L}^{(2)}(u+w_2)\right)$$
$$\sim u\sum_{i=1}^{N}\left(\mu_i\,E_{ii}^{(1)}+n_i\right)+\sum_{i,j=1}^{N}E_{ij}^{(1)}a_j^{\dagger}a_i+\sum_{i=1}^{N}\left(w_2\,E_{ii}^{(1)}+w_1\,n_i\right) \tag{27}$$

$$t(u) = tr\left(\mathcal{L}^{(1)}(u+w_1)\,\mathcal{L}^{(2)}(u+w_2)\right)$$
$$\sim u\sum_{i=1}^{N}\left(\mu_i\,n_{bi}+\nu_i\,n_{ai}\right)+\sum_{i,j=1}^{N}b_i^{\dagger}a_j^{\dagger}a_ib_j+\sum_{i=1}^{N}\left(w_1\,n_{bi}+w_2\,n_{ai}\right) \tag{28}$$

two by two realizations

$$L(u) = \begin{pmatrix} u + \frac{1}{2} S_z & S_+ \\ S_- & u - \frac{1}{2} S_z \end{pmatrix} \quad (29)$$

where,

$$\begin{aligned} [S_z, S_\pm] &= \pm S_\pm \\ [S_+, S_-] &= 2S_z. \end{aligned} \quad (30)$$

$$\mathcal{L}(u) = \begin{pmatrix} u + n_1 & a_1^\dagger a_2 \\ a_2^\dagger a_1 & u + n_2 \end{pmatrix} \tag{31}$$

$$\Lambda(u) = \begin{pmatrix} u + n & \beta a^\dagger \\ \beta a & \beta^2 \end{pmatrix} \qquad \widehat{\Lambda}(u) = \begin{pmatrix} -\beta^2 & \beta a^\dagger \\ \beta a & u - n \end{pmatrix} \tag{32}$$

Also, for $su(1,1)$, one has,

$$L^K(u) = \begin{pmatrix} u + \eta K_z & \eta K_- \\ -\eta K_+ & u - \eta K \end{pmatrix} \tag{33}$$

where,

$$\begin{aligned} [K_z, K_\pm] &= \pm K_\pm \\ [K_+, K_-] &= -2K_z . \end{aligned} \tag{34}$$

Schwinger realizations such as, e.g.,

$S_+ = a\,,\;\; S_- = a^\dagger\,,\;\; S_z = \frac{1}{2}\,(\mathbb{I} - \hat{n})$ or,

$K_+ = a^\dagger b^\dagger\,,\;\; K_- = ab\,,\;\; K_z = \frac{1}{2}\,(\hat{n}_a + \hat{n}_b + 1)$,

may be useful in this context.

Spin Boson model 1

For this model one takes:

$$t(u) = trL(u + w_1)\,\Lambda(u + w_2).$$

Which gives,

$$t(u) \equiv u(\frac{1}{2}\,S_z + n) + w_1(\frac{1}{2}\,S_z + n) + (\alpha\,S_+ a + \alpha\,S_- a^\dagger) + \frac{1}{2} S_z\,n$$

leading to Hamiltonian $H = t(0)$ with conserved quantity

$$I = n + \frac{1}{2} S_z \tag{35}$$

- $a^\dagger$ (a) denotes the single-mode field creation (annihilation) operator.
- $S_z, S_\pm$ the atomic population inversion (imbalance), raising and lowering operators
- w_1 is the transition frequency.

This model describes the interaction of a two-level atom with a single-mode radiation field. It is related to the Jaynes-Cummings Hamiltonian which can be experimentally realized in cavity-QED setups and, as an effective interaction in laser cooled trapped ions.

The shift automorphism (essentially if $L(u)$ is a T matrix so is $L(u+\xi)$, for any ξ) produces a boundary term

$$H_{bound} \quad = \quad \beta\left(\frac{1}{2}S_z(a^\dagger + a) + \alpha\,(S_+ + S_-) + w_1(a^\dagger + a)\right) \tag{36}$$

The conserved quantity changes to

$$I' \quad = \quad n + \frac{1}{2}S_z + \beta\,(a^\dagger + a) \tag{37}$$

Spin boson model 2

Here one takes:

$$t(u) = trL(u + w_1)\,\mathcal{L}(u + w_2).$$

Which gives

$$t(u) \equiv u(n_1 + n_2) + \frac{1}{2}S_z(n_1 - n_2) + S_+ a_1^\dagger a_2 + S_- a_2^\dagger a_1 + w_1(n_1 + n_2)$$

the associated Hamiltonian is,

$$H = t(0) = \frac{1}{2}S_z(n_1 - n_2) + S_+ a_1^\dagger a_2 + S_- a_2^\dagger a_1 + w_1(n_1 + n_2) \quad (38)$$

with conserved quantities

$$I_1 = n_1 + \frac{1}{2}S_z \ ; \ I_2 = n_2 - \frac{1}{2}S_z \quad (39)$$

a_j, $j = 1, 2$ represent two radiation fields (e.g., photons) interacting with a two-level atom. The interaction term may be viewed as the scattering of two fields with a two-level atom.

When the oscillators are bosonic, one can use the shift automorphism to add a boundary term:

$$
\begin{aligned}
H_{bound} &= \alpha_1\Big(\frac{1}{2}S_z(a_1^\dagger + a_1 + \alpha_1) + S_+a_2 + S_-a_2^\dagger + w_1(a_1^\dagger + a_1)\Big) \\
&+ \alpha_2\Big(\frac{1}{2}S_z(a_2^\dagger + a_2 + \alpha_2) + S_+a_1 + S_-a_1^\dagger + w_1(a_2^\dagger + a_2)\Big) \\
&+ \alpha_1\,\alpha_2\,(S_+ + S_-) \qquad (40)
\end{aligned}
$$

The conserved quantities then become

$$
I_1' = n_1 + \frac{1}{2}S_z + \alpha_1\,(a_1^\dagger + a_1) \qquad (41)
$$

$$
I_2' = n_2 - \frac{1}{2}S_z + \alpha_2\,(a_2^\dagger + a_2) \qquad (42)
$$

heteroatomic-molecular model 1

In this case we take:

$$t(u) = tr\mathcal{L}^{(1)}(u + w_1)\,\mathcal{L}^{(2)}(u + w_2).$$

Here, $a_j, a_j^\dagger$, $j = 1, 2$ for $\mathcal{L}^{(1)}$ and $b_j, b_j^\dagger$, $j = 1, 2$ for $\mathcal{L}^{(2)}$.
Which gives,

$$t(u) \sim u(n_1+n_2)+n_{a1}\,n_{b1}+n_{a2}\,n_{b2}+a_1^\dagger b_2^\dagger b_1 a_2+a_2^\dagger b_1^\dagger b_2 a_1+w_1\,n_b+w_2\,n_a$$

the notation used is

$$n_{a1} = a_1^\dagger a_1 \;;\; n_{b1} = b_1^\dagger b_1 \;;\; n_{a2} = a_2^\dagger a_2 \;;\; n_{b2} = b_2^\dagger b_2 \tag{43}$$

$$n_1 = n_{a1} + n_{b1} \;;\; n_2 = n_{a2} + n_{b2} \tag{44}$$

$$n_a = n_{a1} + n_{a2} \;;\; n_b = n_{b1} + n_{b2}\,. \tag{45}$$

From there one has Hamiltonian $H = t(0)$, and conserved quantities

$$I_1 = n_1 \ ; \ I_2 = n_2 \tag{46}$$

This Hamiltonian is a particular case of the next one to be presented. It is also possible to show that the following are conserved quantities:

$$n_1 = n_{a1} + n_{b1} \ ; \ n_2 = n_{a2} + n_{b2} \ ; \ n_a = n_{a1} + n_{a2} \ ; \ n_b = n_{b1} + n_{b2}$$

three being independent.

Heteroatomic-molecular BEC

Here:

$$t(u) = tr\mathcal{L}(u + w_1)\,\Lambda(u + w_2),$$

with $a, a^\dagger$ and $b, b^\dagger$ for $\mathcal{L}(u)$ and $c, c^\dagger$ for $\Lambda(u)$. Which gives,

$$t(u) \equiv u(n_a + n_c) + n_a\,n_c + \beta^2\,n_b + \beta\,(b^\dagger c^\dagger a + a^\dagger bc) + w_1\,n_c + w_2\,n_a$$

$t(0)$ is the Hamiltonian and the conserved quantities are:

$$I_1 = n_a + n_b\ ;\ \ I_2 = n_a + n_c \tag{47}$$

This Hamiltonian is the Heteroatomic- molecular Bose-Einstein condensate model (Foerster, Links, Zhou 2003)

$$\begin{aligned} \mathcal{H} \ &= \ U_{aa}\,n_a^2 + U_{bb}\,n_b^2 + U_{cc}\,n_c^2 + U_{ab}\,n_a\,n_b + U_{ac}\,n_a\,n_c + U_{bc}\,n_b\,n_c \\ &+ \ \mu_a\,n_a + \mu_b\,n_b + \mu_c\,n_c + \Omega\,(a^\dagger bc + c^\dagger b^\dagger a)\,, \end{aligned} \tag{48}$$

for the particular choice of the couplings $\frac{1}{2}U_{aa} = U_{bb} = U_{cc} = U_0 + U_1,\ U_{ab} = 4\,U_0,$
$U_{bc} = 0,\ U_{ac} = 4\,U_1 + 1,\ \mu_a = w_2,\ \mu_b = \beta^2, \mu_c = w_1,\ \Omega = \beta.$

- The U_{ij} describe S-wave scattering.
- μ_i are external potentials
- Ω is the amplitude for interconversion of atoms and molecules.
- three-mode Hamiltonian describing a Bose-Einstein condensate with two distinct species of atoms, denoted b and c, which can combine to produce a molecule a.
- The total atom number $(I_1 + I_2)$ and the imbalance between the atomic modes $(I_1 - I_2)$ are conserved quantities.
- A detailed classical and quantum analysis of this model reveals unexpected scenarios, such as the emergence of quantum phases when the imbalance is zero.

The two-site Bose-Hubbard model

For the 2-site BH model:

$$t(u) = tr\Lambda^{[1]}(u + w_1)\,\Lambda^{[2]}(u + w_2),$$

with $a, a^\dagger$ for $\Lambda^{[1]}$ and $b, b^\dagger$ for $\Lambda^{[2]}$.
One then has

$$t(u) \equiv u(n_a + n_b) + n_a\, n_b + \omega\,(b^\dagger a + a^\dagger b) + w_1\, n_b + w_2\, n_a$$

with $w = \alpha\beta$. The Hamiltonian is given by $t(0)$ with conserved quantity

$$I = n_a + n_b\,. \tag{49}$$

From which,

$$\mathcal{H} = \frac{K}{8}(n_a - n_b)^2 - \frac{\Delta\mu}{2}(n_a - n_b) - \frac{\mathcal{E}_{\mathcal{J}}}{2}(a^\dagger b + b^\dagger a). \quad (50)$$

This is the two-site Bose-Hubbard model, also known as the canonical Josephson Hamiltonian.

- It describes the tunneling between two single particle states or modes (a and b), (two wells) or (two different internal quantum numbers).
- The parameter K corresponds to the atom-atom interaction,
- $\Delta\mu$ is the external potential
- $\mathcal{E}_{\mathcal{J}}$ is the coupling for the tunneling.

Notwithstanding its apparent simplicity, this model possesses a rich structure, e.g. the existence of a threshold coupling between a delocalized and self-trapped phase in qualitative accord with experiments.

2 atoms 'a' and 1 'b'

By a different choice of the Schwinger realization it is possible to find additional models(Santos et al. JPA 2008). Using the following

$$S^+ = b^\dagger c, \quad S^- = c^\dagger b, \quad S^z = \frac{N_b - N_c}{2},$$

$$K^+ = \frac{(a^\dagger)^2}{2}, \quad K^- = \frac{(a)^2}{2}, \quad K^z = \frac{2N_a + 1}{4},$$

One can check that the Hamiltonian

$$\begin{aligned} H &= U_{aa}N_a^2 + U_{bb}N_b^2 + U_{cc}N_c^2 + U_{ab}N_aN_b + U_{ac}N_aN_c + U_{bc}N_bN_c \\ &+ \mu_a N_a + \mu_b N_b + \mu_c N_c + \Omega(a^\dagger a^\dagger b^\dagger c + c^\dagger b a a). \end{aligned} \tag{51}$$

is related with the transfer matrix $t(u)$ through

$$H = t(u) - \frac{1}{2}u^- \eta + \alpha I_1^2 + \beta I_2^2 + \delta I_1 I_2,$$

where $\mu_a = u^- \eta, \quad \mu_c = -\mu_b = u^+ \eta, \quad \Omega = \frac{\eta^2}{2}.$

3 different atoms

Using the following realizations for the $su(2)$ and $su(1,1)$ algebras

$$S^+ = c^\dagger d, \quad S^- = d^\dagger c, \quad S^z = \frac{N_c - N_d}{2},$$

$$K^+ = \frac{(a^\dagger b^\dagger)}{2}, \quad K^- = \frac{(ab)}{2}, \quad K^z = \frac{N_a + N_b + 1}{2},$$

The Hamiltonian

$$\begin{aligned} H &= \alpha I_1^2 + \beta I_2^2 + \delta I_3^2 + \gamma I_1 I_2 + \rho I_1 I_3 + \theta I_2 I_3 \\ &+ \mu_a N_a + \mu_b N_b + \mu_c N_c + \mu_d N_d + \Omega(a^\dagger b^\dagger c^\dagger d + d^\dagger cba) \end{aligned} \quad (52)$$

is related with the transfer matrix $t(u)$ through

$$H(u) = t(u) - u^- \eta + \alpha I_1^2 + \beta I_2^2 + \delta I_3^2 + \gamma I_1 I_2 + \rho I_1 I_3 + \theta I_2 I_3,$$

for the parameters $\mu_a = \mu_b = u^- \eta, \quad \mu_c = -\mu_d = -u^+ \eta, \quad \Omega = \eta^2.$