Solvable Models in Ultracold Physics II

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Contents

Fermion gas

We show first how to build the Bethe ansatz for the two body problem and then we apply the same principles for the three fermions system. The generic N case can be dealt with in the same way.

N=2

Let us then consider two fermions interacting through a delta function potential in a one dimensional system, which has the following Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_2^2} + 2c\delta\left(x_1 - x_2\right). \tag{1}$$

where x_1 and x_2 are the position of each fermion and c is the interaction strength, repulsive for $c > 0$ and attractive for $c < 0$. The action of the Hamiltonian on the ansatz that we are seeking is

$$H\psi\left(x_1, x_2\right) = E\psi\left(x_1, x_2\right), \tag{2}$$

By considerations of continuity and requiring that some terms must vanish, certain consistency relations known as the Bethe ansatz equations will be satisfied.

Energy and momentum are given respectively by

$$E \sim k_1^2 + k_2^2, \;\; K = k_1 + k_2. \tag{3}$$

If we consider the scattering of two particles, then from energy and momentum conservation laws we get

$$E \sim k_1^2 + k_2^2 = k_1^{'2} + k_2^{'2}, \tag{4}$$
$$K = k_1 + k_2 = k_1^{'} + k_2^{'}. \tag{5}$$

the possible solutions are

$$k_1^{'} = k_1, \; k_2^{'} = k_2 \;\; \text{or} \qquad (6)$$
$$k_1^{'} = k_2, \; k_2^{'} = k_1$$

Notice that if one has three particles one has the following 6 particular solutions

$$\{k_1', k_2', k_3'\} = \mathcal{P}\{k_1, k_2, k_3\}, \tag{7}$$

for the equations,

$$k_1^2 + k_2^2 + k_3^2 = k_1'^2 + k_2'^2 + k_3'^2, \tag{8}$$

$$k_1 + k_2 + k_3 = k_1' + k_2' + k_3'. \tag{9}$$

Each of the above solutions is a reflection of another one. But in the case of three or more particles there are solutions that are not in this set. They are called *diffracted* solutions. A generic solution is:

$$
\begin{aligned}
k_1' =& 1/2(k_1 + k_2 - k_3' + k_3 - \\
& \sqrt{(}k_1^2 - 2k_1k_2 + k_2'^2 + 2k_1k_3' + 2k_2k_3' - \\
& 3k_3'^2 - 2k_1k_3 - 2k_2k_3 + 2k_3'k_3 + k_3^2)), \\
k_2' =& 1/2(k_1 + k_2 - k_3' + k_3 + \\
& \sqrt{(}k_1^2 - 2k_1k_2 + k_2'^2 + 2k_1k_3' + 2k_2k_3' - \\
& 3k_3'^2 - 2k_1k_3 - 2k_2k_3 + 2k_3'k_3 + k_3^2)).
\end{aligned} \tag{10}
$$

So, the outgoing waves will be either reflections or diffractions. This is Bethe's hypothesis, so to say, when one has only reflected waves the solution of the many-body Schrödinger equation is reduced to an algebraic one.

Now, the Bethe ansatz, for 1 spin up and 1 spin down, may be written as

$$\psi(x_1,x_2) = \left[a_{12}^{12}e^{i(k_1x_1+k_2x_2)} + a_{21}^{12}e^{i(k_2x_1+k_1x_2)}\right]\Theta(x_2-x_1) + \left[a_{12}^{21}e^{i(k_1x_2+k_2x_1)} + a_{21}^{21}e^{i(k_2x_2+k_1x_1)}\right]\Theta(x_1-x_2)\,. \tag{11}$$

Let us then apply the differential operators of Eq.1 to the above ansatz (consider $\frac{\hbar^2}{2m}=1$).

$$\left(-\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2}\right)\psi(x_1,x_2) = (k_1^2+k_2^2)\psi(x_1,x_2) + e^{i(k_1+k_2)x_1}[2i\delta(x_1-x_2)(k_1-k_2)\left(a_{12}^{12}-a_{21}^{12}+a_{12}^{21}-a_{21}^{21}\right) - \delta'(x_1-x_2)\left(a_{12}^{12}+a_{21}^{12}-a_{12}^{21}-a_{21}^{21}\right)]. \tag{12}$$

Continuity of the ansatz at $x_1=x_2$ tells us that

$$a_{12}^{12}+a_{21}^{12}=a_{12}^{21}+a_{21}^{21}.$$

So, in order to have an eigenvalue equation we need the following:

$$a_{12}^{12} + a_{21}^{12} - a_{12}^{21} - a_{21}^{21} = 0 \tag{13}$$

$$2i(k_1 - k_2)\left(a_{12}^{12} - a_{21}^{12} + a_{12}^{21} - a_{21}^{21}\right) + 2c\left(a_{12}^{12} + a_{21}^{12}\right) = 0,$$

The above can be translated into,

$$A_{12} = \begin{pmatrix} \upsilon_{21} - 1 & \upsilon_{21} \\ \upsilon_{21} & \upsilon_{21} - 1 \end{pmatrix} A_{21} \tag{14}$$

that can be rewritten as:

$$A_{12} = [(\upsilon_{21} - 1)\mathbb{I} + \upsilon_{21}\mathbb{P}_{12}]A_{21}, \tag{15}$$

$$\upsilon_{21} := \frac{k_2 - k_1}{k_2 - k_1 + ic}$$

$$A_{12} := \begin{pmatrix} a_{12}^{12} \\ a_{12}^{21} \end{pmatrix} \quad A_{21} := \begin{pmatrix} a_{21}^{12} \\ a_{21}^{21} \end{pmatrix}$$

$\mathbb{I}$ is the identity operator and $\mathbb{P}_{12}$ is a permutation operator such that

$$\mathbb{P}_{12} \begin{pmatrix} a_{12}^{12} \\ \\ a_{12}^{21} \end{pmatrix} \rightleftarrows \begin{pmatrix} a_{12}^{21} \\ \\ a_{12}^{12} \end{pmatrix} \tag{16}$$

A possible solution is

$$A_{12} = \begin{pmatrix} k_1 - \lambda - i\frac{c}{2} \\ \\ -k_2 + \lambda - i\frac{c}{2} \end{pmatrix} \quad A_{21} = \begin{pmatrix} -k_2 + \lambda + i\frac{c}{2} \\ \\ k_1 - \lambda + i\frac{c}{2} \end{pmatrix} \tag{17}$$

N=3

The three particle case can be dealt with in an analogous way. In the absolute coordinates system it is possible to write the Hamiltonian for three interacting fermions as

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_2^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_3^2} + 2c\delta\left(x_1 - x_2\right) + 2c\delta\left(x_1 - x_3\right) + 2c\delta\left(x_2 - x_3\right) \tag{18}$$

In the region $x_1 < x_2 < x_3$, one can write the wavefunction as

$$\psi\left(x_1, x_2, x_3\right) = a^{123}_{123} e^{i(k_1 x_1 + k_2 x_2 + k_3 x_3)} + a^{123}_{132} e^{i(k_1 x_1 + k_3 x_2 + k_2 x_3)} + a^{123}_{213} e^{i(k_2 x_1 + k_1 x_2 + k_3 x_3)} + a^{123}_{231} e^{i(k_2 x_1 + k_3 x_2 + k_1 x_3)} + a^{123}_{312} e^{i(k_3 x_1 + k_1 x_2 + k_2 x_3)} + a^{123}_{321} e^{i(k_3 x_1 + k_2 x_2 + k_1 x_3)}. \tag{19}$$

Now the vector A is written as

$$A_{123} := \begin{pmatrix} a^{123}_{123} \\ a^{132}_{123} \\ a^{321}_{123} \\ a^{213}_{123} \\ a^{231}_{123} \\ a^{312}_{123} \end{pmatrix} \quad A_{213} := \begin{pmatrix} a^{123}_{213} \\ a^{132}_{213} \\ a^{321}_{213} \\ a^{213}_{213} \\ a^{231}_{213} \\ a^{312}_{213} \end{pmatrix} \tag{20}$$

One has [1,2] :

$$A_{123} = Y^{12}_{21} A_{213}, \tag{21}$$

with

$$Y^{12}_{21} = \frac{i(k_2 - k_1)\mathbb{P}_{12} + c}{i(k_2 - k_1) - c}. \tag{22}$$

The general form of the operator Y is

$$Y_{jl}^{pq} = \frac{i(k_j - k_l)\mathbb{P}_{pq} + c}{i(k_j - k_l) - c}. \tag{23}$$

More explicitly:

$$Y_{21}^{12} = \begin{pmatrix} \upsilon_{21} - 1 & 0 & 0 & \upsilon_{21} & 0 & 0 \\ 0 & \upsilon_{21} - 1 & 0 & 0 & 0 & \upsilon_{21} \\ 0 & 0 & \upsilon_{21} - 1 & 0 & \upsilon_{21} & 0 \\ \upsilon_{21} & 0 & 0 & \upsilon_{21} - 1 & 0 & 0 \\ 0 & 0 & \upsilon_{21} & 0 & \upsilon_{21} - 1 & 0 \\ 0 & \upsilon_{21} & 0 & 0 & 0 & \upsilon_{21} - 1 \end{pmatrix} \tag{24}$$

$$\upsilon_{jl} = \frac{k_j - k_l}{k_j - k_l + ic}$$

Now, it is possible to go from A(123) to A(321) in two possible ways

$$A_{123} \mapsto A_{213} = Y_{12}^{12} A_{123} \mapsto A_{231} = Y_{13}^{23} A_{213} \mapsto A_{321} = Y_{23}^{12} A_{231} \tag{25}$$

$$A_{123} \mapsto A_{132} = Y_{23}^{23} A_{123} \mapsto A_{312} = Y_{13}^{12} A_{132} \mapsto A_{321} = Y_{12}^{23} A_{312}$$

For the equivalence of both paths one must have

$$Y_{23}^{12} Y_{13}^{23} Y_{12}^{12} = Y_{12}^{23} Y_{13}^{12} Y_{23}^{23}. \tag{26}$$

Similarly it is also possible to show $Y_{ij}^{p,p+1} Y_{ji}^{p,p+1} = \mathbb{I}$.

Some definitions:

$$P := \begin{pmatrix} 1 & 2 & \cdots & l & l+1 & \cdots & N \\ P_1 & P_2 & \cdots & P_l & P_{l+1} & \cdots & P_N \end{pmatrix} \quad (27)$$

$$P(l, l+1) := \begin{pmatrix} 1 & 2 & \cdots & l & l+1 & \cdots & N \\ P_1 & P_2 & \cdots & P_{l+1} & P_l & \cdots & P_N \end{pmatrix} \quad (28)$$

Analogously for a Q and then a_P^Q.
For the 3-body case $1 = \uparrow \;\; 2 = \uparrow$ and $3 = \downarrow$

$$\psi\left(x_1, x_2, x_3\right) = -\psi\left(x_2, x_1, x_3\right)$$

$$a_P^{123} = -a_P^{213},\; a_P^{312} = -a_P^{321},\; a_P^{132} = -a_P^{231}$$

$$X_P^1 = \epsilon(P) a_P^{132} \;\; X_P^2 = \epsilon(P) a_P^{321} \text{ and } X_P^3 = \epsilon(P) a_P^{123}$$

where P even (odd) gives $\epsilon = 1\,(-1)$

Using arguments as those of the preceding section we arrive at the following equations,

$$\begin{aligned} X_P^i &= X_{P(l,l+1)}^i \,;\; i \neq l,\, l+1 \\ X_P^j &= (1-\upsilon_{P_j P_{j+1}})X_{P(j,j+1)}^j + \upsilon_{P_j P_{j+1}} X_{P(j,j+1)}^{j+1} \\ X_P^{j+1} &= (1-\upsilon_{P_j P_{j+1}})X_{P(j,j+1)}^{j+1} + \upsilon_{P_j P_{j+1}} X_{P(j,j+1)}^{j}, \end{aligned} \quad (29)$$

that have a solution as,

$$\begin{aligned} X_P^1 &= \eta(k_{P_2} - \Lambda - i\frac{c}{2})(k_{P_3} - \Lambda - i\frac{c}{2}) \\ X_P^2 &= \eta(k_{P_1} - \Lambda + i\frac{c}{2})(k_{P_3} - \Lambda - i\frac{c}{2}) \\ X_P^3 &= \eta(k_{P_1} - \Lambda + i\frac{c}{2})(k_{P_2} - \Lambda + i\frac{c}{2}) \end{aligned} \quad (30)$$

The general N case with 1 spin down and all others up was solved by McGuire, for M=1 down spins and N-2 up by Lieb and Flicker, arbitrary M down spins and N-M up by Gaudin and Yang.
Extending the notation for N-1 up and one down McGuire solution is

$$X_P^{\iota} = \eta \prod_{j=1}^{\iota-1}(k_{P_j} - \Lambda + i\frac{c}{2}) \prod_{l=\iota+1}^{N}(k_{P_l} - \Lambda - i\frac{c}{2}) \equiv \eta F_P^{\iota}(\Lambda).$$

For M down spins and N-M, one has a generalized BA,

$$X_P^{\iota_1,\iota_2,\cdots,\iota_M} = \sum_R \epsilon(R) \prod_{j<l}(\Lambda_{R_j} - \Lambda_{R_l} - ic) F_P^{\iota_1}(\Lambda_{R_1}) F_P^{\iota_2}(\Lambda_{R_2}) \cdots F_P^{\iota_M}(\Lambda_{R_M}).$$

R here is defined as Q, P for the $1, 2, \cdots, M$.

In the case of periodic boundary conditions

$$\psi(x_1, x_2, \cdots, x_N) = \psi(x_N - L, x_1, x_2, \cdots, x_{N-1}) \tag{31}$$

For 3 particles periodic b.c. imply

$$\begin{aligned} e^{ik_1 L} A_{123} &= R_{21} R_{31} A_{123} \\ e^{ik_2 L} A_{123} &= R_{32} R_{12} A_{123} \\ e^{ik_3 L} A_{123} &= R_{13} R_{23} A_{123} \end{aligned} \tag{32}$$

where $R_{ij} = P_{ij} Y_{ij}^{ij}(k_i - k_j)$.

For M and N-M this amounts to another eigenvalue problem (Yang)

$$e^{ik_iL}\,\xi = R_{i+1,i}R_{i+2,i}\cdots R_{N,i}R_{1,i}R_{2,i}\cdots R_{i-1,i}\,\xi, \quad i=1,\cdots,N \tag{33}$$

Yang proposed a generalized Bethe ansatz (hypothesis). Extending our previous notation the above condition may be expressed by (for all permutations),

$$X_P^{i_1,i_2,\cdots,i_M} e^{ik_{P_N}L} = X_{P_N,P_1,\cdots,P_{N-1}}^{i_1+1,i_2+1,\cdots,i_M+1} \tag{34}$$

The solution is

$$e^{ik_jL} = \prod_{\alpha=1}^{M}\left(\frac{k_j-\Lambda_\alpha+i\frac{c}{2}}{k_j-\Lambda_\alpha-i\frac{c}{2}}\right),\ j=1,\cdots,N,$$

$$\prod_{j=1}^{N}\left(\frac{\Lambda_\alpha-k_j+i\frac{c}{2}}{\Lambda_\alpha-k_j-i\frac{c}{2}}\right) = -\prod_{j=1}^{M}\left(\frac{\Lambda_\alpha-\Lambda_\beta+i\frac{c}{2}}{\Lambda_\alpha-\Lambda_\beta-i\frac{c}{2}}\right), \quad \alpha=1,\cdots,M. \tag{35}$$

General results so far

- Coordinate Bethe ansatz
- Yang-Baxter Equation

$$R_{12}(x)R_{13}(x+y)R_{23}(y) = R_{23}(y)R_{13}(x+y)R_{12}(x),$$

$$x = k_2 - k_1,\ y = k_3 - k_2.$$

- Bethe ansatz equations give the quasimomenta k_i and the energy spectrum

$$E = \frac{\hbar^2}{2m}\sum_{j=1}^{N} k_j^2.$$

[1] M. Takahashi, *Thermodynamics of One-Dimensional Solvable Models, Cambridge University Press, 1999*
[2] C. H. Gu and C. N. Yang, *Commun. Math, Phys. 122 (1989)*

QISM

Let us examine Eq.33

$$e^{ik_i L}\,\xi = R_{i+1,i}R_{i+2,i}\cdots R_{N,i}R_{1,i}R_{2,i}\cdots R_{i-1,i}\,\xi,$$
$$i = 1,\cdots,N$$

This kind of eigenvalue equation appears in different settings. A standard example is the Heisenberg model, also called XXX model,

$$H = \frac{J}{2}\sum_{i=1}^{n}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \sigma_i^z\sigma_{i+1}^z\right). \tag{36}$$

Recall the Pauli matrices,

$$\sigma^x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ \sigma^y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\ \sigma^z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{37}$$

Here we take as the permutation operator

$$P_{ij} = \frac{1}{2}\left(1 + \vec{\sigma}_i \cdot \vec{\sigma}_j\right) \tag{38}$$

and

$$R_{ij} = \frac{k_j - k_i + icP_{ij}}{k_j - k_i + ic}. \tag{39}$$

Let us introduce an auxiliary space V_0 such that,

$$P_{i0} = \frac{1}{2}\left(1 + \vec{\sigma}_i \cdot \vec{\tau}_0\right), \tag{40}$$

where τ are also Pauli matrices.
We also define,

$$L_i(k) := \frac{k - k_i + icP_{i0}}{k - k_i + ic}, \tag{41}$$

$$T(k) := L_1(k)L_2(k)\cdots L_N(k).$$

It is easily seen that $L_i(k_i) = P_{i0}$ and one can show that

$$P_{i0}P_{j0} = P_{j0}P_{ij}, \tag{42}$$

such that

$$L_i(k)P_{j0} = \frac{k - k_i + icP_{i0}}{k - k_i + ic}P_{j0} = P_{j0}\frac{k - k_i + icP_{ij}}{k - k_i + ic} = P_{j0}R_{ij}(k_j - k). \tag{43}$$

Now, taking the trace of T over the auxiliary space V_0 we get,

$$\begin{aligned} Tr\{T(k_i)\} &= Tr\{L_1(k_i)L_2(k_i)\cdots L_{i-1}(k_i)L_i(k_i)L_{i+1}(k_i)\cdots L_N(k_i)\} \\ &= Tr\{L_1(k_i)L_2(k_i)\cdots L_{i-1}(k_i)P_{i0}L_{i+1}(k_i)\cdots L_N(k_i)\} \\ &= Tr\{L_{i+1}(k_i)\cdots L_N(k_i)L_1(k_i)L_2(k_i)\cdots L_{i-1}(k_i)P_{i0}\} \\ &= Tr\{P_{i0}\}R_{i+1,i}(k_{i+1} - k_i)R_{i+2,i}(k_{i+2} - k_i)\cdots R_{N,i}(k_N - k_i) \\ &\quad \times R_{1,i}(k_1 - k_i)R_{2,i}(k_2 - k_i)\cdots R_{i-1,i}(k_{i-1} - k_i) \end{aligned} \tag{44}$$

So the eigenvalue problem was rephrased into,

$$Tr\{T(k_i)\}\,\xi = e^{ik_iL}\,\xi. \tag{45}$$

We know from our results above that the operators $t(k_i) := Tr\{T(k_i)\}$ commute

$$[t(k_i), t(k_j)] = 0 \tag{46}$$

A relation that is at the heart of the QISM, along with the Yang-Baxter equation with R from Eq.39,

$$R_{ij}(x)R_{ik}(x+y)R_{jk}(y) = R_{jk}(y)R_{ik}(x+y)R_{ij}(x) \tag{47}$$

RLL

One can rederive the commutation relation for t in a different way, by multiplying the YBE from the left by P_{jk} we get,

$$\begin{aligned} P_{jk}R_{ij}(x)R_{ik}(x+y)R_{jk}(y) &= P_{jk}R_{jk}(y)R_{ik}(x+y)R_{ij}(x) \\ \Rightarrow R_{ik}(x)R_{ij}(x+y)P_{jk}R_{jk}(y) &= P_{jk}R_{jk}(y)R_{ik}(x+y)R_{ij}(x) \end{aligned} \tag{48}$$

In the above if we consider j, k indices for auxiliary spaces V_0 and $V_{0'}$ and redefining the variables by taking Eq.43 into account, we can rewrite it as,

$$L_i(y) \otimes L_i(x)R(x-y) = R(x-y)L_i(x) \otimes L_i(y) \tag{49}$$

This is sometimes called Yang-Baxter relation and R is the matrix

$$R(x-y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{-ic}{x-y-ic} & \frac{x-y}{x-y-ic} & 0 \\ 0 & \frac{x-y}{x-y-ic} & \frac{-ic}{x-y-ic} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{50}$$

Now calculating

$$\begin{aligned} & R(x-y)\{T(x)\otimes T(y)\}R^{-1}(x-y) \\ & = [L_1(y)\otimes L_1(x)][L_2(y)\otimes L_2(x)]\cdots[L_N(y)\otimes L_N(x)] \\ & = T(y)\otimes T(x), \end{aligned} \tag{51}$$

for x and y respectively equal k_i and k_j and taking the trace in the auxiliary space the commutation relation for t is obtained again.
Eq.51 gives as well the $RTT = TTR$ relation.
Some names:

- $R \rightarrow R$-matrix.
- $L \rightarrow$ Lax operator.
- $T \rightarrow$ Monodromy matrix.
- $t \rightarrow$ Transfer matrix.

Solving the Eigenvalue Eq.

The generic structure of the monodromy matrix is

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \tag{52}$$

Its entries contain Pauli matrices acting on the spin variables of the chain.

$$T(u) \otimes T(v) = \begin{pmatrix} A(u)A(v) & A(u)B(v) & B(u)A(v) & B(u)B(v) \\ A(u)C(v) & A(u)D(v) & B(u)C(v) & B(u)D(v) \\ C(u)A(v) & C(u)D(v) & D(u)A(v) & D(u)B(v) \\ C(u)C(v) & C(u)D(v) & D(u)C(v) & D(u)D(v) \end{pmatrix}$$

$RTT = TTR$ imply 16 commutation relations among A, B, C, D the relevant ones for our purpose are

$$B(u)B(v) = B(v)B(u) \tag{53}$$

$$B(v)A(u) = \frac{v-u}{v-u-ic}A(u)B(v) - \frac{ic}{v-u-ic}B(u)A(v) \tag{54}$$

$$B(v)D(u) = \frac{ic}{v-u+ic}B(u)D(v) + \frac{v-u}{v-u+ic}D(u)B(v) \tag{55}$$

The state where all spins are up is denoted $|0\rangle$,

$$|0\rangle = |\uparrow\uparrow\cdots\uparrow\rangle \tag{56}$$

The operator B acting on $|0\rangle$, produces a state with one spin down, A and C annihilate $|0\rangle$, and this state is an eigenvalue of D. What we want is to find the eigenvalues of $t(u) = Tr[T(u)] = A(u) + D(u)$, such that $t(k)\xi = e^{ikL}\xi$. The proposed algebraic Bethe ansatz is

$$\prod_{\alpha=1}^{M} B(v_\alpha)\,|0\rangle \tag{57}$$

Applying $t(u)$ on the ansatz gives,

$$[A(u) + D(u)] \prod_{\alpha=1}^{M} B(v_\alpha)\,|0\rangle = \tag{58}$$

$$\left[\prod_{\gamma=1}^{M} \frac{1}{b(v_\gamma - u)} + \frac{\prod_{r=1}^{N} b(u-k_r)}{\prod_{\gamma=1}^{M} b(u-v_\gamma)}\right] \prod_{\alpha=1}^{M} B(v_\alpha)\,|0\rangle +$$

$$\sum_{\beta=1}^{M} \frac{c(v_\beta - u)}{b(v_\beta - u)} \left[\prod_{\gamma\neq\beta}^{M} \frac{1}{b(v_\gamma - v_\beta)} + \frac{\prod_{r=1}^{N} b(v_\beta - k_r)}{\prod_{\gamma\neq\beta}^{M} b(v_\beta - v_\gamma)}\right] B(u) \prod_{\alpha\neq\beta}^{M} B(v_\alpha)\,|0\rangle$$

So we have something like

$$t\,\xi = (\text{eigenvalue})\,\xi + \text{unwanted terms}. \tag{59}$$

Requiring the cancelation of the unwanted terms gives

$$\prod_{r=1}^{N} b(v_\beta - k_r) = \prod_{\gamma\neq\beta} \frac{b(v_\beta - v_\gamma)}{b(v_\gamma - v_\beta)}, \tag{60}$$
$$1 \leqslant \beta \leqslant M,$$

and for $u = k_r$ we get the condition,

$$e^{ik_r L} = \prod_{\gamma=1}^{M} \frac{1}{b(v_\gamma - k_r)}, \tag{61}$$

where $b(u) = u/(u - ic)$ and $c(u) = -ic/(u - ic)$.
These are again the Bethe ansatz equations by the QISM.