Solvable Models in Ultracold Physics I

Itzhak Roditi

Centro Brasileiro de Pesquisas Físicas
Rio de Janeiro, Brasil

August, 2014

Contents

Why study ultracold physics?

- In high temperature random motion essentialy dominates. So we are, in the case of gases, at essentialy a classical limit. The behavior is much that of classical particles colliding with each other.
- In ultracold temperatures quantum effects becomes relevant.

○ Wave behavior is no longer negligible.

◇ Quantum coherent phenomena occur.

◇ Such as, superfluidity, superconductivity or Bose-Einstein condensation (BEC).

◇ Coherence is also related to: entanglement, quantum information processing, and quantum metrology.

- Quantum phase transitions.

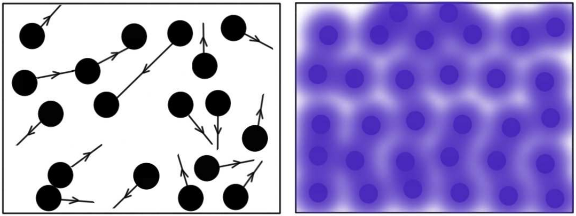


Figure: Left Hot —– Right Cold

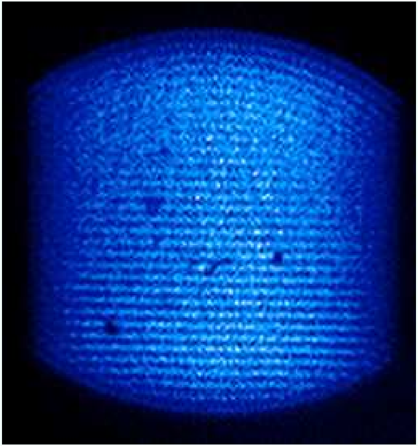


Figure: Berylium ions ultracold plasma NIST experiment, temperature measurements of ultracold crystals. A "plasma" of tens of thousands of singly charged beryllium atoms is trapped using electric and magnetic fields and then cooled to almost absolute zero using lasers. When the lasers are turned off, the plasma begins to heat up. At 10 milliKelvin—just 0.01 degree above absolute zero—the temperature suddenly rises more than 10 billion times faster than predicted by theory. This burst of energy in a very cold system of highly interactive particles is designed to simulate events occurring inside the hot, dense interiors of stars, where plasmas of highly charged atoms undergo accelerated nuclear reactions.

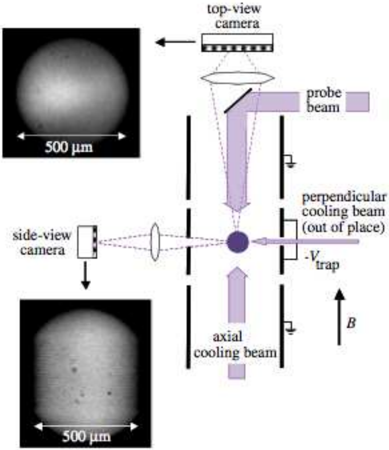
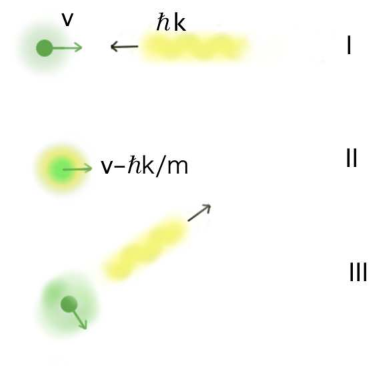


FIG. 1 (color online). Schematic diagram of setup. Figure is not to scale. The trap diameter is 4 cm. Top- and side-view images of a plasma with 26 000 ions are shown. Individual crystalline planes are visible in the side-view image. The vertical edges of the plasma visible in the side-view image are due to the presence of nonfluorescing impurity ions heavier than $^9\mathrm{Be}^+$.

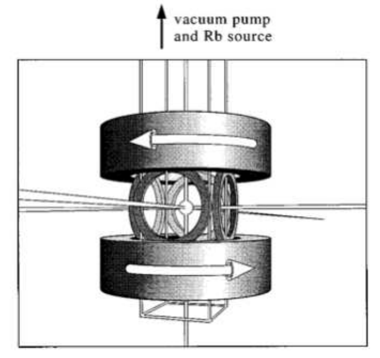
Reaching cold temperatures



I - Atom velocity $= v$ collides with a photon (momentum$= \hbar k$);
II - Atom velocity $= v - \hbar k/m$ after light absorption;
III After re-emission (randomly directed) velocity lowers on average.

This process is exploited in the **Doppler cooling**. One drawback is that it has a limit around mK, which is still quite high. By imposing additional laser beams that could keep the atoms in hyperfine levels, or "optically pumped", lower temperatures where achieved, but still not enough to obtain Bose condensation.
Researchers had then to find other methods to cool even more the atoms ensembles.

Magneto Optic Traps

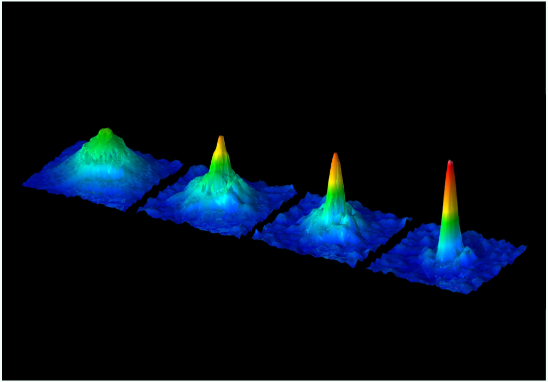


Light from diode lasers comes from all six directions to form a MOT in the middle of the cell. Running current through the magnetic- field coils shown surrounding the cell creates the magnetic trap. Fig. from Wieman et al. Rev.Mod.Phys. 1999.

Evaporative cooling

Further lowering of temperature is achieved by the fact that the more energetic atoms escape the trap.

Bose-Einstein condensate



Velocity distributions of trapped atoms. Colors correspond to the number of atoms at each velocity, blue being the fewest - red being the most.

Why study solvable models?

- They serve as a platform to understand the physics beneath several models, allowing to establish a benchmark, for numerical and analytical calculations.
- One paradigmatic example is the two-dimensional Ising model that is almost "too simple" as a model for ferromagnetism, but has proven to be crucial for many a concept in critical phenomena.
- They often reveals unexpected links with sophisticated mathematical results.
- They now emerge in various experiments.

Ising

It may be rightly said that the two dimensional Ising model for $H = 0$ is one of the most important systems studied in theoretical physics. It is the first statistical mechanical system which can be exactly solved which exhibits a phase transition. From the exact results for the free energy [1], spontaneous magnetization [2],[3] and correlation functions [4]-[8] a point of view has been developed, which embraces the concepts of scaling, universality and conformal field theory, that extends the exact results of the Ising model to more general situations. These concepts are widely used to analyze both experiments and models of critical phenomena. Furthermore the correlation functions provide very concrete realizations of the concepts of mass and wave function renormalization used to define Euclidean quantum field theories.

From: *The importance of the Ising model, Barry M. McCoy and Jean-Marie Maillard*

Ising

It may be rightly said that the two dimensional Ising model for $H = 0$ is one of the most important systems studied in theoretical physics. It is the first statistical mechanical system which can be exactly solved which exhibits a phase transition. From the exact results for the free energy [1], spontaneous magnetization [2],[3] and correlation functions [4]-[8] a point of view has been developed, which embraces the concepts of scaling, universality and conformal field theory, that extends the exact results of the Ising model to more general situations. These concepts are widely used to analyze both experiments and models of critical phenomena. Furthermore the correlation functions provide very concrete realizations of the concepts of mass and wave function renormalization used to define Euclidean quantum field theories.

From: *The importance of the Ising model, Barry M. McCoy and Jean-Marie Maillard*

Ising

It may be rightly said that the two dimensional Ising model for $H = 0$ is one of the most important systems studied in theoretical physics. It is the first statistical mechanical system which can be exactly solved which exhibits a phase transition. From the exact results for the free energy [1], spontaneous magnetization [2],[3] and correlation functions [4]-[8] a point of view has been developed, which embraces the concepts of scaling, universality and conformal field theory, that extends the exact results of the Ising model to more general situations. These concepts are widely used to analyze both experiments and models of critical phenomena. Furthermore the correlation functions provide very concrete realizations of the concepts of mass and wave function renormalization used to define Euclidean quantum field theories.

From: *The importance of the Ising model, Barry M. McCoy and Jean-Marie Maillard*

Ising

It may be rightly said that the two dimensional Ising model for $H = 0$ is one of the most important systems studied in theoretical physics. It is the first statistical mechanical system which can be exactly solved which exhibits a phase transition. From the exact results for the free energy [1], spontaneous magnetization [2],[3] and correlation functions [4]-[8] a point of view has been developed, which embraces the concepts of scaling, universality and conformal field theory, that extends the exact results of the Ising model to more general situations. These concepts are widely used to analyze both experiments and models of critical phenomena. Furthermore the correlation functions provide very concrete realizations of the concepts of mass and wave function renormalization used to define Euclidean quantum field theories.

From: *The importance of the Ising model, Barry M. McCoy and Jean-Marie Maillard*

Ising

$$s_i = \begin{cases} +1 & \text{spin up } \uparrow \\ -1 & \text{spin down } \downarrow \end{cases} \tag{1}$$

$$\begin{array}{cccccc}
\uparrow & \downarrow & \uparrow & \downarrow & \downarrow & \uparrow \\
\uparrow & \downarrow & \downarrow & \uparrow & \uparrow & \downarrow \\
\uparrow & \downarrow & \downarrow & \uparrow & \uparrow & \downarrow \\
\uparrow & \downarrow & \uparrow & {\color{blue}\downarrow} & \downarrow & \uparrow \\
\uparrow & \downarrow & {\color{blue}\downarrow} & {\color{red}\uparrow} & {\color{blue}\uparrow} & \downarrow \\
\uparrow & \downarrow & \downarrow & {\color{blue}\uparrow} & \uparrow & \uparrow \\
\uparrow & \downarrow & \uparrow & \downarrow & \downarrow & \uparrow
\end{array} \tag{2}$$

$$\text{Energy} = -J \sum_{<i,j>} s_i s_j - H \sum_i s_i . \tag{3}$$

Ising

$$\text{Energy} = -J\sum_{<i,j>} s_i s_j - H\sum_i s_i. \tag{4}$$

$J > 0 \Rightarrow$ ferromagnetism: the energy is minimal when the spins point in the same direction . $J < 0 \Rightarrow$ antiferromagnetic: spins are antiparallel . H is an external magnetic field coupled to the magnetization $\sum_i s_i$. When H=0 two different phases exist depending on the temperature . For low temperatures (T), the system is permanently magnetized. At sufficiently high T, the magnetization of the system is zero. There exists a critical value of the temperature, T_c (the Curie temperature) at which there is a phase transition between the ferromagnetic (permanently magnetized) and paramagnetic phases.

Ising

$$\text{Energy} = -J \sum_{<i,j>} s_i s_j - H \sum_i s_i. \tag{5}$$

$J > 0 \Rightarrow$ ferromagnetism: the energy is minimal when the spins point in the same direction . $J < 0 \Rightarrow$ antiferromagnetic: spins are antiparallel. H is an external magnetic field coupled to the magnetization $\sum_i s_i$. When H=0 two different phases exist depending on the temperature. For low temperatures (T), the system is permanently magnetized. At sufficiently high T, the magnetization of the system is zero. There exists a critical value of the temperature, T_c (the Curie temperature) at which there is a phase transition between the ferromagnetic (permanently magnetized) and paramagnetic phases.

Ising

$$\text{Energy} = -J \sum_{<i,j>} s_i s_j - H \sum_i s_i. \tag{6}$$

$J > 0 \Rightarrow$ ferromagnetism: the energy is minimal when the spins point in the same direction . $J < 0 \Rightarrow$ antiferromagnetic: spins are antiparallel . H is an external magnetic field coupled to the magnetization $\sum_i s_i$. When H=0 two different phases exist depending on the temperature . For low temperatures (T), the system is permanently magnetized. At sufficiently high T, the magnetization of the system is zero. There exists a critical value of the temperature, T_c (the Curie temperature) at which there is a phase transition between the ferromagnetic (permanently magnetized) and paramagnetic phases.

Ising

$$\text{Energy} = -J \sum_{<i,j>} s_i s_j \; - \; H \sum_i s_i . \tag{7}$$

$J > 0 \Rightarrow$ ferromagnetism: the energy is minimal when the spins point in the same direction . $J < 0 \Rightarrow$ antiferromagnetic: spins are antiparallel . H is an external magnetic field coupled to the magnetization $\sum_i s_i$. When H=0 two different phases exist depending on the temperature . For low temperatures (T), the system is permanently magnetized. At sufficiently high T, the magnetization of the system is zero. There exists a critical value of the temperature, T_c (the Curie temperature) at which there is a phase transition between the ferromagnetic (permanently magnetized) and paramagnetic phases.

On the same family

$$\textbf{Ising} \to \mathcal{H} = -J \sum_{<i,j>} s_i s_j = -J \sum_{<i,j>} S_i^z S_j^z .$$

$$\textbf{XY} \to \mathcal{H} = J \sum_{<i,j>} \left[S_i^x S_j^x + S_i^y S_j^y \right] .$$

$$\textbf{Heisenberg} \to \mathcal{H} = J \sum_{<i,j>} \left[S_i^x S_j^x + S_i^y S_j^y + S_i^z S_j^z \right] .$$

$$S^x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; S^y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; S^z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (8)$$

Spin chain models

$$\textbf{XXX} \to \mathcal{H} = J \sum_i \left[S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + S_i^z S_{i+1}^z \right].$$

$$\textbf{XXZ} \to \mathcal{H} = J \sum_i \left[S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \right].$$

$$\textbf{XYZ} \to \mathcal{H} = -\sum_i \left[J_x\, S_i^x S_{i+1}^x + J_y\, S_i^y S_{i+1}^y + J_z\, S_i^z S_{i+1}^z \right].$$

These are models of magnetism.

Boson and Fermion models

One dimensional Bose/Fermi gas with δ function (or contact interaction)

$$\mathcal{H} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i<i} \delta(x_i - x_j). \tag{9}$$

Solvable by the Bethe ansatz.
For $c = 0$ in second quantized form in a box:

Bosons $\rightarrow \mathcal{H} = \sum_k k^2 a_k^+ a_k, \qquad \{k\} = \{2\pi\, n/L\}$

Fermions $\rightarrow \mathcal{H} = \sum_k k^2 \left(c_{k\uparrow}^+ c_{k\uparrow} + c_{k\downarrow}^+ c_{k\downarrow}\right), \qquad \{k\} = \{2\pi\, n/L\}$

$$\left[a_k, a_l^+\right] = \delta_{kl}, \quad [a_k, a_l] = \left[a_k^+, a_l^+\right] = 0$$

$$\{c_{ks}^+, c_{ls'}\} = \delta_{kl}\delta_{ss'}, \quad \{c_{ks}, c_{ls'}\} = \{c_{ks}^+, c_{ls'}^+\} = 0$$

Gas in a lattice

One-dimensional **Hubbard model**

$$\mathcal{H} = -t \sum_{i=1}^{L} \sum_{s=\uparrow,\downarrow} \left(c_{is}^{+} \, c_{i+1s} + h.c. \right) + U \sum_{i=1}^{L} n_{i\uparrow} n_{i\downarrow} \tag{10}$$

$$n_{i\,s} = c_{i\,s}^{+} \, c_{i\,s} \,. \tag{11}$$

- It is maybe the most studied model of high temperature superconductivity.
- Model of metallic ferromagnetism.
- Model of quantum (insulating) antiferromagnetism.
- It is solvable in one dimension.

See F. H. L. Essler, H. Frahm, F. Göhmann, A. Klümper and V. E. Korepin, The One- Dimensional Hubbard Model (Cambridge University Press, Cambridge, 2005).

Bose-Hubbard

A related model with bosons instead of fermions

$$\mathcal{H} = \sum_{i=1}^{N} [U\, n_i(n_i - 1)/2 - \mu\, n_i] - t \sum_{i=1}^{N} (a_i^+\, a_{i+1} + h.c.). \qquad (12)$$

solvable, for instance, in the two-site case and also in the hard-core boson limit where it is the analogue of a lattice Tonks-Girardeau gas.

Some experiments

- **QPT in Bose-Hubbard**
  "For a system at a temperature of absolute zero, all thermal fluctuations are frozen out, while quantum fluctuations prevail. These microscopic quantum fluctuations can induce a macroscopic phase transition in the ground state of a many-body system when the relative strength of two competing energy terms is varied across a critical value."(excerpt from Greiner et al. Nature 2002)

In this paper they report an observation of $^{87}\mathrm{Rb}$ Bose-Einstein condensates trapped in a three-dimensional optical lattice. The condensate goes from a superfluid to a Mott insulator gas due to the competition between U and t.

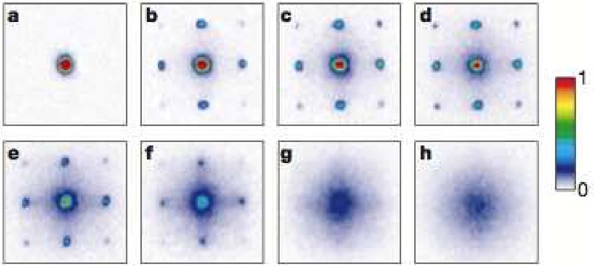


Figure 2 Absorption images of multiple matter wave interference patterns. These were obtained after suddenly releasing the atoms from an optical lattice potential with different potential depths V_0 after a time of flight of 15 ms. Values of V_0 were: **a**, 0 E_r; **b**, 3 E_r; **c**, 7 E_r; **d**, 10 E_r; **e**, 13 E_r; **f**, 14 E_r; **g**, 16 E_r; and **h**, 20 E_r.

Around a potential depth of $13E_r = 13\hbar^2 k^2/2\pi m, \;\; k = 2\pi/\lambda$, coherence is lost. (from Greiner et al.)

- **Hubbard model - Mott phase**
  "Here we report the formation of a Mott insulator of a repulsively interacting two-component Fermi gas in an optical lattice. It is identified by three features: a drastic suppression of doubly occupied lattice sites, a strong reduction of the compressibility inferred from the response of double occupancy to an increase in atom number, and the appearance of a gapped mode in the excitation spectrum."(Jördens et al. Nature 2008)

Because there is no symmetry breaking, there is a crossover from the metallic to the Mott insulating regime at finite temperature rather than a phase transition. Experiments have been realized with $^{40}\mathrm{K}$ atoms (see also Schneider et al. Science 2008).

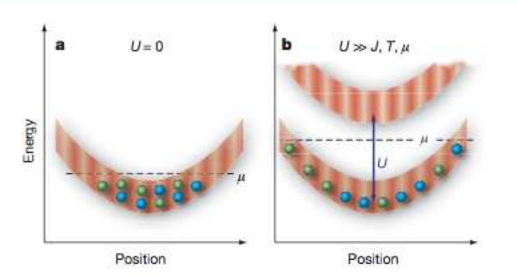


Figure: **a**, In the non-interacting case the curvature of the lowest Bloch band reflects the harmonic confinement. At zero temperature all states up to the chemical potential μ are filled with atoms of both spin states (green and blue). **b**, In the Mott insulating limit the energy cost for creating doubly occupied sites greatly exceeds the temperature T and the kinetic energy parametrized by J, giving rise to a gap of order U. The energy spectrum of single-particle excitations is then depicted by two Hubbard bands. Doubly occupied sites correspond to atoms in the upper Hubbard band. (From Jördens et al.)

Two-site Bose-Hubbard

A model for two condensates coupled by Josephson tunneling.

$$\mathcal{H} = \frac{K}{8}(N_1 - N_2)^2 - \mu(N_1 - N_2) - \frac{\epsilon}{2}(a_1^+ a_2 + a_1 a_2^+) \tag{13}$$

It is a solvable model as we shall see later.

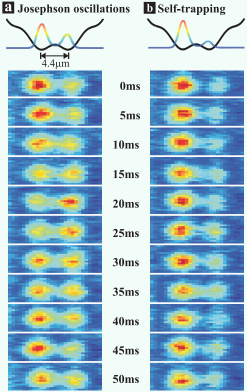


Figure: From: Albiez et al. PRL 2005

Why experimentalists are interested in low dimensions?

- Impressive progress in cooling and trapping allows studies of one-dimensional structures.
- Quantum phase transitions.
- Trapping of few particles.
- Non-equilibrium studies.

Follows a list of key experiments in 1D quantum atomic gases (from Guan, Batchelor and Lee RMP, 2013).

Group (Leader)	Research topics
Amsterdam (van Druten)	Yang-Yang thermodynamics (2008) non-equilibrium spin dynamics (2010)
Cambridge (Köhl)	quantum transport (2009)
CNRS (Bouchoule)	density fluctuations (2006, 2011) phonon fluctuations (2012) 1D-3D crossover (2011) three-body correlations (2010) mean-field breakdown (2006)
ENS (Salomon)	matter-wave solitons (2002)
ETH (Esslinger, Köhl)	confinement induced molecules (2005) p-wave Feshbach resonance (2005) 1D-3D crossover (2004) Bragg spectroscopy (2004) collective oscillations (2003)
Hamburg (Sengstock)	matter-wave solitons (2008)
Innsbruck (Nägerl)	super-Tonks-Girardeau gases (2009) sine-Gordon phase transition (2010) confinement induced resonance (2010) three-body correlations (2011)
Kaiserslautern (Ott)	spatiotemporal fermionization (2012)
LENS (Inguscio)	Bragg spectroscopy (2009, 2011, 2012) low-energy excitations (2009) impurity dynamics (2012)
Mainz/MPQ (Bloch)	impurity dynamics (2013) relaxation dynamics (2012, 2013) squeezed Luttinger liquids (2008) Tonks-Girardeau gases (2004)
MIT (Ketterle)	atomic interferometry (2007) fluctuations and squeezing (2007)
NIST (Phillips, Porto)	dipole oscillations (2005) three-body recombination (2004)
Pennsylvania (Weiss)	Tonks-Girardeau gases (2004) local pair correlations (2005) quantum Newton's cradle (2006)
Rice Uni. (Hulet)	spin-imbalanced Fermi gases (2010) matter-wave solitons (2002)
Vienna (Schmiedmayer)	quantum correlations (2011, 2012) twin-atom beams (2011) atomic interferometry (2005, 2010) quantum and thermal noises (2008) non-equilibrium dynamics (2007)