Observables in solvable lattice models

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Preface for the summerschool LDQS 2014

Also these notes have been prepared for another occasion. But the method presented here is equivalent to that applied some 10 years later to loop models. For this reason I try to present them in one lecture series.

Abstract

Some calculational methods for solvable lattice models in two-dimensional statistical mechanics are presented, and explicitly applied to the eight-vertex model. The Yang-Baxter equation is discussed, and the solution for the eight-vertex model is given. The inversion relation is used to calculate the free energy. A difference equation method presented in 1993 by Jimbo, Miwa and Nakayashiki[4], is explained here, and applied to the calculation of expectation values. In particular the spontaneous staggered polarization of the eight vertex model is treated in detail. Prior knowledge of solvable models or the eight-vertex model is not a prerequisite for the use of these notes.

1. Introduction

The aim of statistical mechanics is the calculation of macroscopic observables from microscopic interactions for systems with many, possibly infinitely many degrees of freedom. In the case of static or equilibrium statistical mechanics this calculation is simply the performance of summations or integrations of the probability distribution. The main difficulty is that for large systems these summations or integrations are (infinitely) many and cannot in general be decoupled. For this reason almost all calculations implement some approximation scheme.

For solvable lattice models the entire calculation of the partition function and of many of the thermodynamic averages can be done exactly for the infinite system, and in some cases also for finite systems as a function of system size. There are of course many models for which this program can be carried out by virtue of some pathology: some aspect in which the model differs in en essential way from real materials. This is for example the case with hierarchical models and models on the Bethe lattice in which the lattice structure is non-periodic. Other example are the spherical model, or models with infinitely weak, infinite range interactions. These pathologies may be acceptable in certain cases in which it can be shown that their effect is numerically insignificant.

However there is a large class of models on regular lattices, with strictly short range interactions, which are solvable without being in any way non-physical or trivial [1]. In particular the parameter space in which these models can be solved in most cases includes

critical points and often the different phases on either side of the transition. In the great majority of cases they are two-dimensional models.

The Boltzmann weight of these solvable models can in most cases be written as a product of factors (or equivalently the Hamiltonian as a sum of terms), each of which depend only on a few variables located one or a few lattice spacings apart. The following classes are generally distinguished. *Edge models* have variables on the sites of a (usually square) lattice and interactions between variables on nearest neighbor sites. *IRF models*, short for *I*nteractions *R*ound a *F*ace, also have variables on each site of the lattice, but the local Boltzmann weights involves the four spins on the corners of an elementary square. *Vertex models* have variables on each edge of the lattice and interactions among the four edges around each site or vertex of the lattice. These three classes of models can be viewed as different representations, since they can be transormed one into the other. The models are usually formulated on the square lattice, but in most cases they can be generalized to other lattices.

Also in three dimensions some classes of solvable models have been found [2, 3], but much less is known about them, and the Boltzmann weights may take negative or complex values.

The main subject of these lectures is a method recently developed by Jimbo *et al.* [4] to calculate correlation functions in solvable vertex models. This method is based on the realization of quantum affine symmetry and the use of vertex operators [5]. The method was demonstrated quite clearly with graphical means by Jimbo *et al.* [4]. In order to make the lectures methodologically self-contained, we will first discuss the Yang-Baxter equation[1], its meaning and solution in the case of the eight-vertex model. The inversion relation method[6] is applied for the calculation of the free energy. Though in principle the methods are general and apply to all solvable models, the attention will be focussed on the eight-vertex model. The main sources for these lectures are [1], [6], [7] and [4]. Especially [4] is followed in detail.

2. Yang-Baxter equation for the eight-vertex model

2.1. The eight-vertex model

The symmetric eight-vertex model on the square lattice, has two-state variables on all the edges of the lattice usually represented by means of arrows pointing along the edge. Not all configurations of these arrows are allowed, only those in which the number of arrows directed into each vertex is even. This rule allows eight arrow configurations around each vertex, hence the name eight-vertex model. These configurations and their Boltzmann weights are shown in figure 1. The weight function, denoted as $w(\alpha, \beta, \gamma, \delta)$, is symmetric for the simultaneous inversion of the arrows, and otherwise general. It takes the values a, b, c and d, as indicated in the figure. Conversely, a, b, c and d are the parameters by which the function w is parametrized. The state of the arrows is also denoted by one of the values 1 (up, right) and -1 (down, left).

For a configuration on the regular square lattice, all vertices thus have the weight a, b, c or d depending on the local arrow configuration, and the total Boltzmann weight of the configuration is the product of all the local vertex weights. The partition sum is then the sum of these products over all possible arrow configurations.

We will be concerned with what is called the principal domain of the phase diagram $c \geq a+b+d$ and all weights non-negative. This is an antiferromagnetically ordered phase in which both horizontal and vertical arrows preferentially alternate. The order parameter to detect this ordering is the staggered polarization of the arrows. It can be shown ([8], and sec.10.2 of [1]) from symmetry considerations and duality, that the free energy in all other regions of the phase diagram can be simply related to that of the principal domain.

2.2. The transfer matrix

One way to calculate the partition sum of a latice N rows and L columns, is to arrange the summation in separate sums over rows of variables, as follows. The configuration of the j-th row of vertical and horizontal arrows are called v_j and h_j respectively, then the partition sum can be written as a sum on all $v_{1,\ldots,N}$ and $h_{1,\ldots,N}$. The Boltzmann weight factorizes in factors that involve only one h_j and two consecutive v_j, so that

$$\begin{aligned} Z &= \sum_{\{v\}} \prod_{j=1}^{N} \sum_{h_j} W(v_{j-1}, h_j, v_j) \\ &= \mathrm{Tr}\, T^N = \sum_j \Lambda_j^N \end{aligned} \tag{1}$$

where $T(v,v') = \sum_h W(v,h,v')$ is called the row-to-row transfer matrix, and the Λ_j are its eigenvalues. Since the eigenvalues are discrete, the partition sum is for sufficiently large N given by the power of the largest eigenvalue, Λ_0^N.

2.3. The Yang-Baxter equation

We will now combine the vertices in a manner that does not normally occur in a regular square lattice. As in fig. 2 we join three vertices together and take the three weight functions to be different, say w, w' and w'', which take the values a, b, c and d in one vertex, a', b', c' and d' in the second and a'', b'', c'' or d'' in the third vertex. The Yang-Baxter (YB) equation now asserts that for suitably chosen weights a through d'' the partition sum of this small lattice is the same when the vertices are combined as in the left or right figure of fig. 2.

$$\sum_{\mu,\nu,\eta} w(\tau,\eta,\nu,\sigma) w'(\alpha,\beta,\mu,\eta) w''(\mu,\gamma,\delta,\nu) = \sum_{\mu,\nu,\eta} w(\nu,\beta,\gamma,\eta) w'(\mu,\eta,\delta,\sigma) w''(\alpha,\nu,\mu,\tau) \tag{2}$$

This equality should hold for all possible configurations of the six external variables, while the three internal variables are summed over. Before we discuss the solution of this

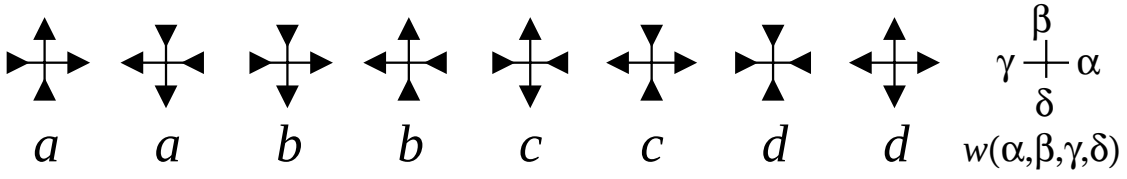

Figure 1. The local states and Boltzmann weights of the eight-vertex model.

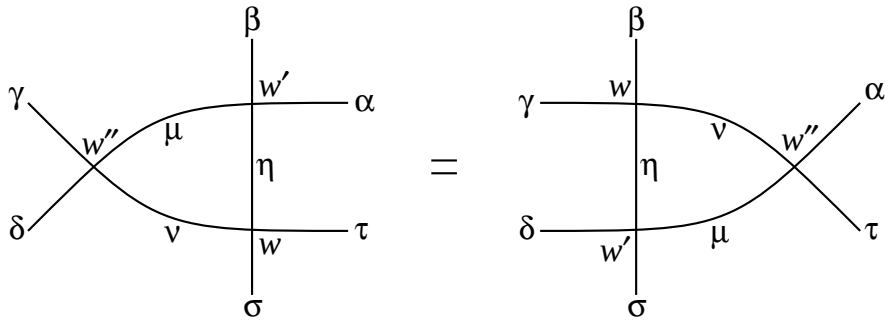

Figure 2. The Yang-Baxter equation for vertex models. The variables on the internal edges are summed over.

equation, we first demonstrate one of the consequences, which serves as a justification of looking into the equation at all.

Suppose we have found a solution w, w', and w'' of the YB equation. Then we construct two transfer matrices, T and T' from the weight functions w and w' respectively. We further suppose that there exists a fourth function w''' which satisfies

$$\sum_{\mu,\nu} w'''(\nu,\tau,\sigma,\mu)w''(\alpha,\nu,\mu,\beta) = \delta_{\alpha,\tau}\delta_{\beta,\sigma} \tag{3}$$

i.e. w''' is the matrix inverse of w''. The product of T and T' can be graphically represented as in fig. 3.a, in which the horizontal external edges on the right and left are identified, so that the system has periodic boundary conditions. Because of eq.(3) the matrix product $w'''w''$ may be inserted without affecting the result: fig. 3.b. Because w, w' and w'' satisfy the YB equation, and these three vertices occur in the same combination as in the left-hand side of fig. 2, the vertex with weight function w'' may be moved one step to the right. This process may be repeated until this vertex is moved all the way to the right and can be combined again with w''', due to the periodic boundary condition. As a result the order of the transfer matrices is reversed, and since all the steps were based on an equality we now have the equation,

$$T'T = TT' \tag{4}$$

This commutation property of transfer matrices based on different vertex weights, is a cornerstone in a variety of methods to calculate eigenvalues of T. An important ingredient in such a calculation is the determination of as many as possible instances of T', that commute with T. For this it is necessary to look into the solution of the YB equation. The commutation of transfer matrices does not play a central role in these lectures, but its derivation serves here as an illustration of the power of the YB equation.

2.4. Solution of the Yang-Baxter equation

The YB equation is a set of algebraic equations on the weights $a \ldots d''$. In section 10.4 of [1] these equations are explicitly written out and solved. We will not repeat the algebra here, but only sketch the main steps. By eliminating the doubly primed weights (in which

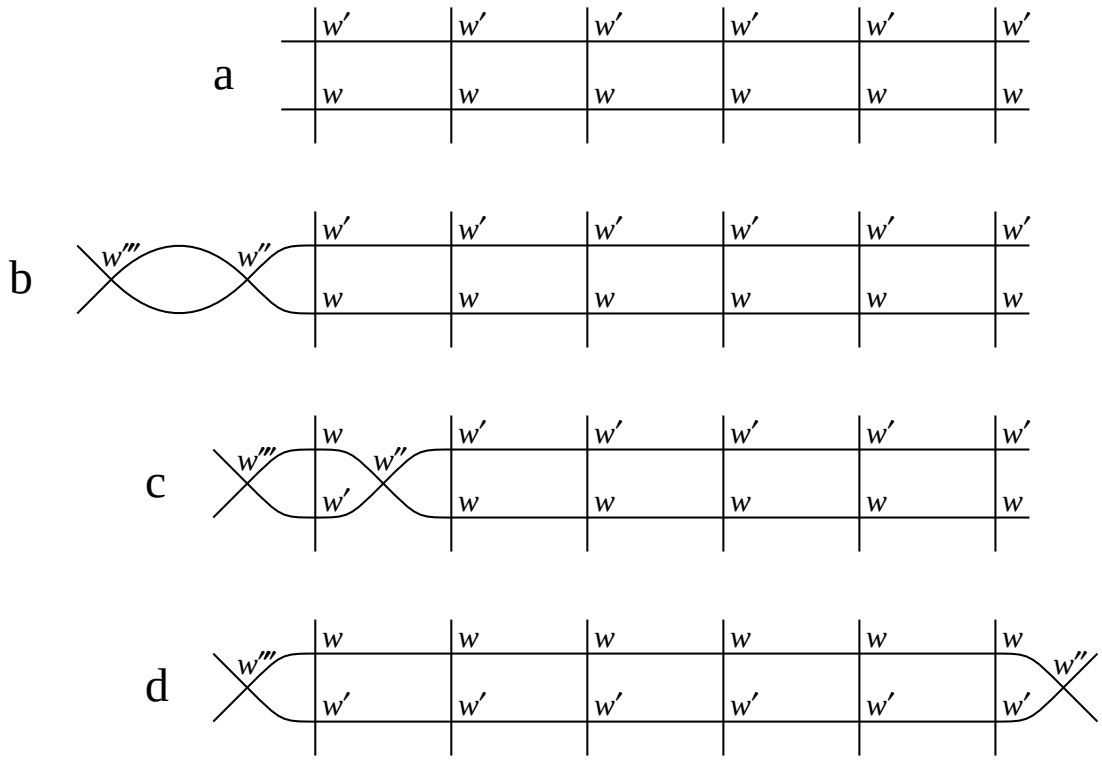

Figure 3. The product of two different transfer matrices of which the local weights w and w' satisfy the YB equation (a). The left- and rightmost horizontal edges are identified. The edges internal to the figure must be summed over. The product matrix product $w''w'''$ may be inserted (b). Due to the YB equation w'' may be moved to the right (c). When w'' is moved all the way to the right, it may be combined again with w''' (d), and the result is that the transfer matrices are commuted.

the equations are linear), one finds the conditions on w and w' that permit a non-zero w''. These conditions can be phrased simply as

$$\Gamma' = \Gamma \quad \text{and} \quad \Delta' = \Delta \tag{5}$$

in which

$$\begin{aligned} \Gamma &= (ab - cd)/(ab + cd) \\ 2\Delta &= (a^2 + b^2 - c^2 - d^2)/(ab + cd) \end{aligned} \tag{6}$$

and the same definitions with all symbols primed. From the symmetry of the YB equation it is obvious that also $\Gamma'' = \Gamma$ and $\Delta'' = \Delta$.

The weight function w and therefore the transfer matrix T is parametrized by the four parameters a, b, c and d. A rescaling of all four has no physical effect, so there are besides an overal normalization three relevant parameters. Equation (5) indicates that it is possible to reparametrize the weights such w and w' agree in two of the parameters and differ in only one.

Therefore we now wish to rewrite the weights in terms of parameters, say ρ, λ, q and u, such that a, b, c and d are entire functions of u (*i.e.* analytic in the entire complex u-plane), Γ and Δ are independent of u, and ρ is an overal normalization. The rationale is the following. Since transfer matrices with different u commute, the eigenvalues are fixed linear combinations of the matrix elements, and they too must be entire functions of u. This is a powerful tool in determining their precise form. Also in other methods analyticity plays a central role, as we will see below.

In [1] Baxter introduced a parametrization based on elliptic functions, that satisfies the above requirements.

$$\begin{aligned} a &= -\mathrm{i}\rho\Theta(\mathrm{i}\lambda)H(\mathrm{i}\lambda-\mathrm{i}u)\Theta(\mathrm{i}u), \qquad & b &= -\mathrm{i}\rho\Theta(\mathrm{i}\lambda)\Theta(\mathrm{i}\lambda-\mathrm{i}u)H(\mathrm{i}u), \\ c &= -\mathrm{i}\rho H(\mathrm{i}\lambda)\Theta(\mathrm{i}\lambda-\mathrm{i}u)\Theta(\mathrm{i}u), \qquad & d &= \quad \mathrm{i}\rho H(\mathrm{i}\lambda)H(\mathrm{i}\lambda-\mathrm{i}u)H(\mathrm{i}u), \end{aligned} \tag{7}$$

The functions Θ and H are defined as

$$\begin{aligned} H(u) &= 2q^{1/4}\sin\frac{\pi u}{2I}\prod_{n=1}^{\infty}(1-2q^{2n}\cos\frac{\pi u}{I}+q^{4n})(1-q^{2n}) \\ \Theta(u) &= \prod_{n=1}^{\infty}(1-2q^{2n-1}\cos\frac{\pi u}{I}+q^{4n-2})(1-q^{2n}) \end{aligned} \tag{8}$$

The *nome* q is related to the *half-period magnitude* I by

$$I=\frac{\pi}{2}\prod_{n=1}^{\infty}\left(\frac{(1+q^{2n-1})(1-q^{2n})}{(1-q^{2n-1})(1+q^{2n})}\right)^2 \tag{9}$$

The normalization ρ does not affect the equations or the physics, and will be set to unity. The other parameters may be real or complex. When all are chosen real, also the weights are real-valued, but for instance when both u and λ are imaginary, the weights are also real (up to an overal factor). For notational convenience we introduce the variables

$$\begin{aligned} x &= \exp\left(\frac{-\pi\lambda}{2I}\right) \\ z &= \exp\left(\frac{-\pi u}{2I}\right) \end{aligned} \tag{10}$$

Since we will occasionally still use λ and u, we will always keep this fixed relation between λ and u on one hand and x and z on the other. The principal domain $c > a+b+d$ is given by $0<q<\exp(\frac{-\pi\lambda}{I})<\exp(\frac{-\pi u}{I})<1$, or equivalently $0<q<x^2<z^2<1$.

In these variables eq.(5) is simplified to the requirement that the parameters x and q are the same for the weights w and w', while z may be different. The weight function w'' now has the same form as the other two, with $u''=u-u'$, or equivalently $z''=z/z'$

With the parameter dependence made explicit, the consequence of commuting transfer matrices now reads

$$T_{z,x,q}\,T_{z',x,q} \;=\; T_{z',x,q}\,T_{z,x,q} \tag{11}$$

Thus, the transfer matrix of an eight-vertex model is a member of a family of commuting transfer matrices, parametrized by a single (complex) parameter z. For this reason z, or

more commonly u is called the *spectral parameter*, as it determines the spectrum (without affecting the eigenvectors). We will mainly deal with the dependence on this parameter, while x and q are kept fixed. Therefore also the dependence on x and q will be suppressed in the sequel, while the dependence on the spectral parameter z will be indicated as a suffix.

Under the transformation $z \to x/z$ the weights a and b are interchanged, while c and d remain as they are, *i.e.* $a_z = b_{x/z}$, $c_z = c_{x/z}$ and $d_z = d_{x/z}$. Due to this so called crossing symmetry, the parameter x is called *crossing parameter*. At $z = 1$ the weights $b_{z=1} = d_{z=1} = 0$ vanish, while $a_{z=1} = c_{z=1}$. Thus the local Boltzmann weight viewed as a matrix, acts like a multiple of the identity, from the left and bottom arrows to the right and top arrows. Likewise for $z = x$, the weights $a_{z=x} = d_{z=x} = 0$ vanish, while $b_{z=x} = c_{z=x}$.

Another important property of the local Boltzmann weight is the inversion relation

$$\sum_{\mu,\nu} w_z(\alpha,\beta,\nu,\mu) w_{1/z}(\mu,\nu,\tau,\sigma) = \delta_{\alpha,\sigma}\delta_{\beta,\tau} g_z \tag{12}$$

in which $g_z = h(\lambda+u)h(\lambda-u)$ and $h(u) = \mathrm{i}\Theta(\mathrm{i}u)H(\mathrm{i}u)\Theta(0)$. This equality can be verified from the elliptic function identities (15.4.19) and (15.4.20) of [1].

3. The partition sum per site κ_z

3.1. Inversion relation for the diagonal transfer matrix

Consider a square lattice drawn diagonally, as in fig. 4. A transfer matrix can be defined again by ordering the summation of the partition sum into sums over configurations of horizontal rows of variables. If the edge variables are enumerated horizontally, $\alpha_1 \ldots \alpha_L$ with L even and $\alpha_{L+1} \equiv \alpha_1$, we may introduce the operator U_j with matrix elements

$$(U_j)_{\alpha,\beta} = w\left(\beta_{j+1}, \beta_j, \alpha_j, \alpha_{j+1}\right) \prod_{k \neq j, j+1} \delta_{\alpha_k,\beta_k} \tag{13}$$

The diagonal-to-diagonal transfer matrix $T^{(d)}$ acts as the following two operators in succession

$$\begin{aligned} V &= U_2 U_4 U_6 \cdots U_{L-2} U_L \\ W &= U_1 U_3 U_5 \cdots U_{L-3} U_{L-1} \\ T^{(d)} &= VW \end{aligned} \tag{14}$$

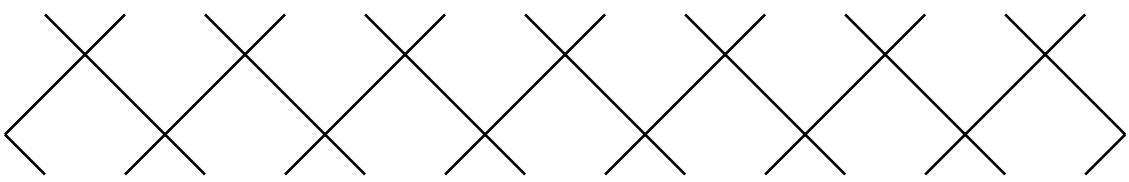

Figure 4. A section of a diagonally drawn lattice, corresponding with $T^{(d)}$. The external edges at the upper and lower edge constitute the indices of the matrix, the vertices at the extreme left and right are identified, and the internal edges are summed over.

Note that the even numbered and the odd numbered U commute among themselves, but not with each other. An immediate consequence of (12) is that $V_z V_{1/z} = W_z W_{1/z} = g_z^{L/2}$. Since the eigenvalues Λ_z of the matrix $T^{(d)} = V_z W_z$ are the same as those of $W_z V_z$, they must satisfy

$$\Lambda_z \Lambda_{1/z} = g_z^L. \tag{15}$$

The partition sum per vertex is related to the largest eigenvalue by $\kappa_z = \Lambda_z^{1/L}$, therefore one expects

$$\kappa_z \kappa_{1/z} = g_z. \tag{16}$$

However, if all eigenvalues at $1/z$ are inversely proportional to those at z, the largest must be thus related to the smallest. Since in the end we want the largest eigenvalue only in the physical regime, $x < z < 1$, we will just accept in the continuation to larger z the smallest eigenvalue. With that qualification (16) is valid.

Since the weights (7) are periodic in $\log(z)$, with period $2\pi\mathrm{i}$, so must be κ_z. Of course this periodicity has already been implied by using z as the spectral parameter rather than u. In fact since it can be shown that in each configuration on an even sized lattice with periodic boundary conditions each of the weights a, b, c and d occurs an even number of times, κ_z will have the period of the square of the weights, which is $\pi\mathrm{i}$.

Under the transformation $z \to x/z$ the weights a and b are simply interchanged. Since the corresponding arrow configurations differ only by a horizontal or vertical reflection, the free energy remains unaltered under this operation:

$$\kappa_z = \kappa_{x/z} \tag{17}$$

3.2. Solution of κ_z

The properties (16), (17) and the periodicity are almost sufficient to determine κ_z, but we must take one bold step. We assume that κ_z is analytic and non-zero for $x - \epsilon < |z| < 1 + \epsilon$, for some positive ϵ. This can be made plausible by inspecting the low temperature expansion for κ, which gives no sign of a singularity in this annulus in the complex plane.

Periodicity and analyticity imply that the free energy has a Laurent expansion

$$\log \kappa_z = \sum_{k=-\infty}^{\infty} f_k z^{2k} \tag{18}$$

convergent in the annulus $x - \epsilon < |z| < 1 + \epsilon$. We now take $1 - \epsilon < |z| < 1 + \epsilon$ so that z , $1/z$, and x/z all lie in the domain of analyticity, so that we can apply (16) and (17). Because of the product form of these equations and the defining elliptic functions it is straightforward to take the logarithm of left- and right hand side, expand the result in powers of z, and equate the coefficients of the same power. The resulting equations for the coefficients

$$\begin{aligned} f_0 &= -\log x + \frac{1}{4}\log q + \sum_{n=1}^{\infty} \log\left(1 - q^n\right) + 2\log\left(1 - q^{2n}\right) \\ f_k + f_{-k} &= -\frac{x^{2k} + q^k x^{-2k}}{k(1 - q^k)} \\ f_{-k} &= x^{2k}\, f_k \end{aligned} \tag{19}$$

can be solved trivially, yielding for the free energy

$$\log \kappa_z = f_0 - \sum_{k=1}^{\infty} \frac{\left(x^{2k} + q^k x^{-2k}\right)\left(z^{2k} + x^{2k} z^{-2k}\right)}{k\,(1-q^k)\,(1+x^{2k})} \tag{20}$$

For future use it is convenient to write κ_z as an infinite product of algebraic factors. This can be accomplished by expanding the denominators $(1-q^k)$ and $(1+x^{2k})$ under the summation. Subsequent performing of the sum on k results in a double sum of logarithmic terms, which by exponentiating of both sides of the equation result in an infinite product of algebraic terms. Obviously the prefactor coming from f_0 requires only the last step. In order to denote the result compactly we introduce the following function

$$(z; q_1, \ldots, q_k)_\infty = \prod_{n_1=0}^{\infty} \cdots \prod_{n_k=0}^{\infty} (1 - z q_1^{n_1} \ldots q_k^{n_k}) \tag{21}$$

The partition sum per vertex can then be written

$$\kappa_z = x^{-1} q^{1/4} \frac{f(z)}{f(z^{-1})} \left(x^2 z^{-2}; q\right)_\infty \left(q x^{-2} z^2; q\right)_\infty (q;q)_\infty^2 \left(q^2; q^2\right)_\infty , \tag{22}$$

in which the function $f(z)$ is given by

$$f(z) = \left(x^2 z^2; q, x^4\right)_\infty \left(q x^2 z^2; q, x^4\right)_\infty \left(x^4 z^{-2}; q, x^4\right)_\infty \left(q z^{-2}; q, x^4\right)_\infty . \tag{23}$$

That (22) is a solution of (16) and (17) can of course be verified directly by substitution. That the solution is unique, however, follows from the unique solution of (19) for a converging Laurent expansion, of which the existence rests upon the assumption of analyticity.

4. Z-invariance

4.1. General planar lattices

The goal of the procedures is to calculate thermodynamic functions for the eight vertex model on a regular lattice. Nevertheless we now consider (see [7]) a general planar lattice consisting of intersecting lines of which no three lines intersect in the same vertex. For the moment it suffices to assume that the lines are straight. We consider a convex region of the plane, as show in fig. 5. With every interior edge, or line segment bounded by two intersections, we associate a two-state arrow variable, that may point from one to the other vertex. The exterior edges, that is the line segments bounded by an intersection and the boundary will be occupied by a fixed arrow variable. With each intersection we associate an eight-vertex weight function parametrized by q, x and z, such that all q and x are the same, but the z may vary from vertex to vertex.

We will require that for all sets of three intersections between three lines, the values of z are chosen such that the order of the intersections may be reversed by means of the YB equations. Baxter observed [7] that this may be accomplished by the introduction of z variables associated with the lines rather than the vertices, such that the spectral parameter of an intersection between the lines j and k is given by the ratio z_j/z_k of the

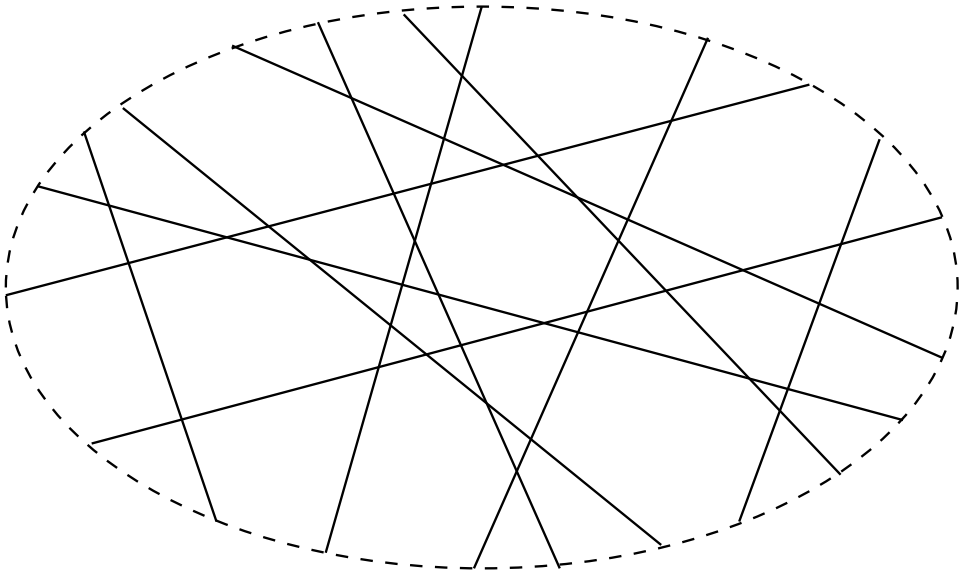

Figure 5. An example of an irregular lattice consisting of intersecting lines. The boundary is shown as a dashed line.

variables associated with the lines. It is then necessary to give an orientation to each of the lines, in order to set the convention which ratio should be taken: z_j/z_k or z_k/z_j.

In the figures the orientation of the lines will be indicated with open arrows, distinct from the solid arrows denoting the states of the dynamical variables. To fix the convention we let the spectral parameter of an intersection of a downward oriented vertical line with z_1 and a left oriented horizontal line with z_2 be given by z_1/z_2, see fig. 8b below. Other cases may be formed from this one by continuous rotation of the lines. This also determines the way the Boltzmann weights are assigned to the local vertex state. When all four state arrows agree with the orientation of the line on which they live (or when all are opposite), the weight is a, when they agree on one line and are opposite to the orientation on the other the weight is b. When on one line both arrows point in and on the other out, the weight is c, and when all arrows point in or all out the weight is d.

These conventions guarantee that each triplet of intersections of three lines has values of the (vertex) spectral parameter that conform to the left-hand side or right-hand side of the YB equation, so that this equation can be invoked to shift one of the lines across the intersection of the other two. However, this is true only if the line orientation of the three lines is not cyclic, *i.e.* it must not be possible to follow the inner triangle consisting of the three edges between the three intersections cyclicly in the direction of the three line orientations, see fig. 6. This restriction can be satisfied for all sets of three lines by demanding that the orientation of all lines projected onto a single reference line be the same. Provided cyclic orientations of intersecting lines do not occur, it is no longer necessary to require that the lines are or remain straight.

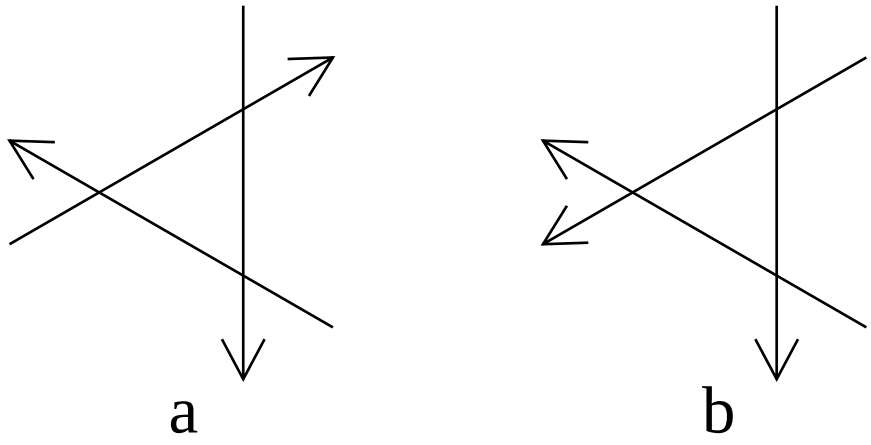

Figure 6. Cyclic (a) and non-cyclic (b) line orientations. The Boltzmann weights at the intersections of cyclicly oriented lines, do not satisfy the YB equation. Thus line orientations should be chosen such that no cyclic combinations occur.

4.2. Invariance of partition sum and correlations

The lines forming the lattice may now be moved. Moves that do not alter the topology of the lattice obviously have no effect at all. Whenever a line is shifted across the intersection of two others, the YB equation guarantees that this also will have no effect on the partition sum, Z, of the lattice. For this reason such changes are called Z-invariant moves. But the invariance is stronger than that. When three intersections thus change their order, three edge variables disappear, and three new variables are created. As long as these variables are summed over, all correlations among the other variables are unaffected. We will use the phrase Z-invariance in this generalized sense as also applicable to correlations. This invariance has far reaching consequences, which we will discuss below.

Let us assume that we have a very large lattice, in which the interior correlations are unaffected by what happens far away on the boundary. Now consider the correlation between two edges deep inside the lattice, $\langle \varepsilon_j \varepsilon_k \rangle$. All movements of lines by repeated application of the YB equations, that leaves the edges j and k intact, also leave their correlation invariant. That implies that the correlation can depend only on the spectral parameters of the lines of which the edges j and k are taken and the lines that intervene between these two edges. All other lines with their spectral parameters can be moved away arbitrarily far, where they are not expected to have an effect. Since the correlation is not changed by these moves, the lines had no effect to begin with. This argument can of course be repeated for correlations involving any number of edges.

4.3. Local thermodynamic properties

The partial derivative of the free energy with respect to the spectral parameter of a single vertex v is a correlation involving only the arrow variables of that vertex, say $\langle \sigma_v \rangle$. If we vary the line spectral parameters z_j the variation of the free energy δf can be written

as a sum over all vertices v,

$$\delta f = \sum_v \langle \sigma_v \rangle \, \delta \left(z_j / z_k \right), \tag{24}$$

in which j and k are the lines that intersect in v. According to the argument above, $\langle \sigma_v \rangle$ can depend only on z_j/z_k, and not on any of the other spectral parameters, nor on the topology of the lattice, provided of course that v is not near any boundary. As a consequence the variation of the free energy with the spectral parameters is a sum of functions of the parameters of a single vertex. Then the free energy itself must also have this form, and according to this argument [7] the partition sum is the product over all vertices

$$Z = \prod_v \kappa_{z_j/z_k} \tag{25}$$

for a lattice which is large enough that the effect of the boundary can be neglected.

4.4. Properties of the normalized Boltzmann weight R.

Because of the form of the partition sum (25) it is meaningful to normalize the Boltzmann weights of each vertex individually

$$R(z)_{\alpha\beta\gamma\delta} = \frac{w_z(\alpha,\beta,\gamma,\delta)}{\kappa_z} \tag{26}$$

so that the partition sum will be normalized to unity, and no normalization is needed for the correlation functions. As a convenient consequence of this normalization (26) the inversion relation (12) now has shed its coefficient g_z.

$$\sum_{\mu,\nu} R(z_1/z_2)_{\alpha,\beta,\nu,\mu} \, R(z_2/z_1)_{\mu,\nu,\tau,\sigma} = \delta_{\alpha,\sigma}\delta_{\beta,\tau}. \tag{27}$$

This equation can be graphically represented as in fig. 7. It permits the removal of two intersections between the same lines, leaving no trace in the partition sum or correlations among edges other than the removed ones. It should be noted that the relative orientation of the two lines must be parallel. Again this restriction is automatically satisfied if all line orientation have a common projection to a reference line.

At the special values $z = 1$ and $z = x$ the matrix R reduces to

$$\begin{aligned} R(1)_{\alpha,\beta,\nu,\mu} &= \delta_{\alpha,\mu}\delta_{\beta,\nu} \\ R(x)_{\alpha,\beta,\nu,\mu} &= \delta_{\alpha,-\beta}\delta_{\mu,-\nu} \end{aligned} \tag{28}$$

On occasion we want to be able to change the orientation of a line. For a reference orientation we take vertical lines to be oriented downward, horizontal lines to the left, fig. 8b. As indicated in fig. 8, the orientation of the lines may be changed, with a corresponding change in the spectral parameter, such that the actual weight for each configuration remains the same. If the vertical line is turned up, while the horizontal points left, its spectral parameter should be divided by x. If the horizontal line is reversed, while the vertical remains down, its spectral parameter should be multiplied by x. The formula to describe the transition of fig. 8b to a

$$R(z_1/z_2)_{\alpha,\beta,\gamma,\delta} = R(xz_2/z_1)_{\alpha,-\delta,\gamma,-\beta} \tag{29}$$

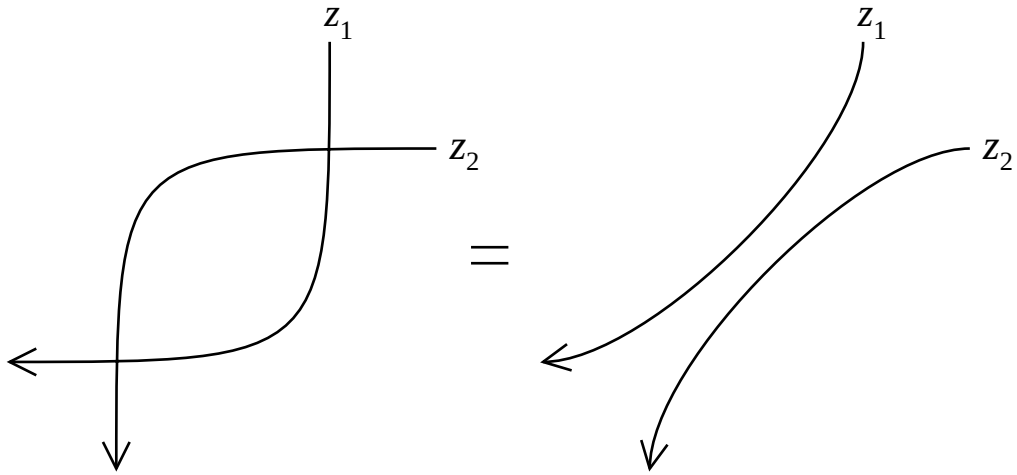

Figure 7. Graphical representation of the inversion relation (27). The two vertices may be completely removed with a Z-invariant move.

is of course an expression of crossing symmetry. A note of caution: the orientation of a line can be changed only if the orientations of the intersecting lines in projection to the orthogonal of the original line are the same. This is clear from comparing the transition from fig. 8b to a and from c to d. In both cases the vertical orientation is changed from down to upward, but the effect on the spectral parameter is opposite, due to the difference in the orientation of the horizontal line. This restriction is alleviated by the fact that there it is possible to change the orientation of only a part of a line.

In the reference orientation fig. 8b, the Boltzmann weights are associated with the configurations as in fig. 1 (up to normalization with κ_z). For other orientations the identifications can be found either by rotating the figure appropriately, or by inverting line orientations and consulting fig. 8. It may be convenient to remember that the Boltzmann weight has the value a/κ of (7) when the state arrows all agree with the orientation of the intersecting lines (or when they all point in the opposite direction).

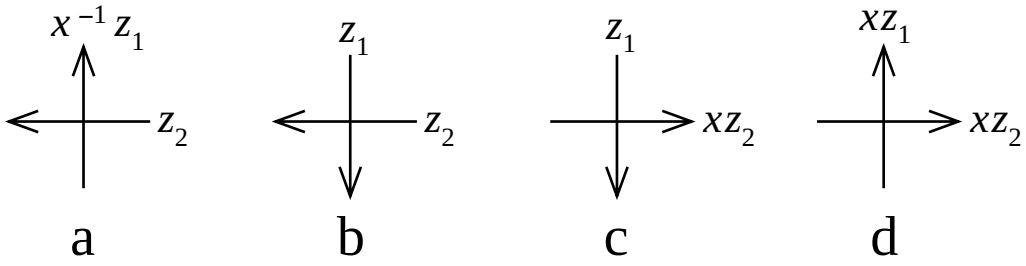

Figure 8. The orientation of the lines may be changed, with appropriate change of the spectral parameter.

5. Correlations and dislocations

5.1. Definitions

We now consider again a square lattice, of which the vertical lines are oriented downward, and the horizontal lines to the left. The spectral parameters of the lines may differ from line to line, and the vertical ones will be referred to as z_j. Jimbo *et al.* have derived difference equations in order to calculate correlation functions involving a horizontal row of consecutive vertical edges. Since any set of edges can be transformed into a subset of a row of vertical edges by means of Z-invariant moves, the results are in principle applicable to correlation functions of arbitrary sets of edge variables.

We are considering the principal domain of the phase diagram, in which the weight c is the largest. In this domain there are two ground state configurations, in which all vertices are in the state with weight c/κ. These two configurations are related by a total spin flip and we may thus distinguish between them by the state of a reference edge. It does not mean that the reference edge will be kept fixed: we suppose that the choice of ground state is enforced by fixing the external edges of the whole lattice. By the state of the reference edge we mean the state it would be in if the ground state compatible with the boundary conditions is enforced on the entire system.

Consider a row of n successive vertical edges, between two neighboring horizontal lines. The edge variables are $\varepsilon_1, \ldots, \varepsilon_n$, and the spectral parameters associated with the corresponding vertical lines are $z_1, \ldots, z_n$, numbered from left to right. The orientation of the vertical lines is downward, and of the horizontal lines to the left. Fig. 9 shows the configuration of the n edges, as enclosed between two horizontal lines, and two vertical lines. The rectangle formed by the two horizontal lines and the vertical lines to the left of ε_1 and to the right of ε_n will be referred to as the *central face*. For the reference edge we take the vertical edge to the left of the variable ε_1. Eventually we want to know the probability that the n edges are in a given state. Since the local Boltzmann weights have

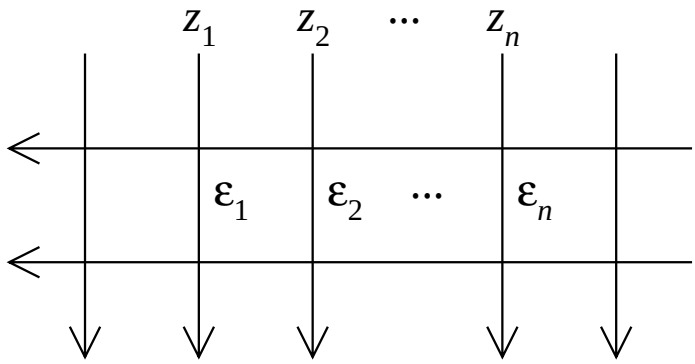

Figure 9. The correlation functions. We consider the probability that the edge variables $\varepsilon_1, \ldots, \varepsilon_n$ are in a specific state. The figure shows only the relevant section of a large square lattice.

been normalized we will assume that these correlations can be replaced by configuration

sums in which the n edge variables are kept fixed. The lattice is taken very large in all directions. The spins on the boundary are also fixed, in a state consistent with one of the two ground state configurations. From the principle of Z-invariance the configuration sums will be independent of all spectral parameters, other than $z_1, \ldots, z_n$, and also of the configuration of the lattice outside the central face. This extremely powerful invariance is the very core of the present approach, and can hardly be emphasized enough.

We will proceed to derive difference equations for a more general class of objects than just these correlation functions. The first step is to cut the n edges, in the sense that they break up in two independent edge variables, and that the spectral parameter above and below the break may be different. The original correlators can of course be found by specifying that the edge variable as well as the spectral parameter above and below the break be equal. The second step is that a dislocation of the lattice is allowed across

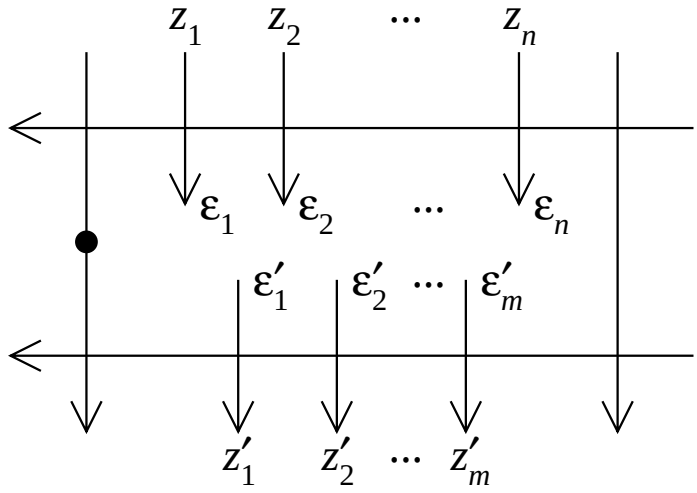

Figure 10. The function $F^{(i)}_{n,m}(z_1, \ldots, z_n; z'_1, \ldots, z'_m)_{\varepsilon_1,\ldots,\varepsilon_n;\varepsilon'_1,\ldots,\varepsilon'_m}$ is the configuration sum in which the edge variables $\varepsilon_1, \ldots, \varepsilon_n$ and $\varepsilon'_1, \ldots, \varepsilon'_m$ are kept fixed. The reference edge is indicated by a black disk.

the break, in the sense that the number of vertical lines entering from above and below may differ. Thus the most general object which we consider is shown in fig. 10, a face of the lattice in which n upper half-lines and m lower half-lines are inserted. With $F^{(i)}_{n,m}(z_1, \ldots, z_n; z'_1, \ldots, z'_m)_{\varepsilon_1,\ldots,\varepsilon_n;\varepsilon'_1,\ldots,\varepsilon'_m}$ we denote the configuration sums in which the upper edge variables are fixed at $\varepsilon_1, \ldots, \varepsilon_n$ and the lower edge variables at $\varepsilon'_1, \ldots, \varepsilon'_m$. The spectral parameters above and below are denoted by z_j and z'_j respectively. The upper index i denotes the state of the reference spin as $(-1)^i$. In practice we will mainly deal with cases in which $m = 0$, denoted by $F^{(i)}_n(z_1, \ldots, z_n)_{\varepsilon_1,\ldots,\varepsilon_n}$. The numbering of the edge variables and spectral parameters is from left to right. The orientation of the vertical lines is downward, and of the horizontal lines to the left.

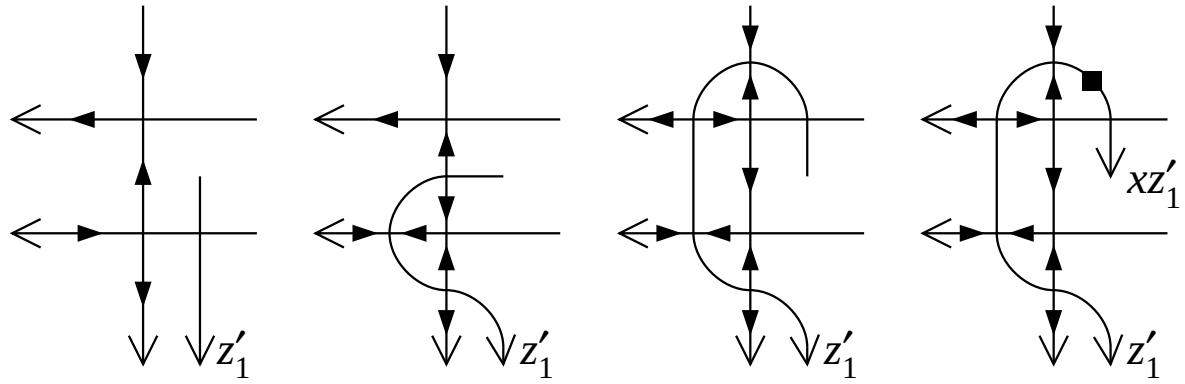

Figure 11. Rotation of the left-most line, eq.(30). Applications of the YB equation and inversion relation are followed by the use of crossing symmetry to invert the orientation of the rotated line from the black square downward. The ground state configuration is indicated to show that the reference edge is inverted.

5.2. Derivation of the functional equations

The lines from above and below can be exchanged by the following equation

$$\begin{aligned} & F^{(i)}_{n,m}(z_1,\dots,z_n;z'_1,\dots,z'_m)_{\varepsilon_1,\dots,\varepsilon_n;\varepsilon'_1,\dots,\varepsilon'_m} \\ &= F^{(i+1)}_{n+1,m-1}(xz'_1,z_1,\dots,z_n;z'_2,\dots,z'_m)_{-\varepsilon'_1,\varepsilon_1,\dots,\varepsilon_n;\varepsilon'_2,\dots,\varepsilon'_m} \\ &= F^{(i)}_{n-1,m+1}(z_1,\dots,z_{n-1};z'_1,\dots,z'_m,xz_n)_{\varepsilon_1,\dots,\varepsilon_n;\varepsilon'_1,\dots,\varepsilon'_m,-\varepsilon'_n} \end{aligned} \tag{30}$$

This is the result of a sequence of Z-invariant moves and the application of crossing symmetry as show in fig. 11. If the left-most line is moved from the lower to the upper edge of the central face, it passes by the reference spin, which is reversed in the process. This does not happen when the right-most line is moved. The edge variable on the rotated line is reversed, because the line is turned upside down. The remnant of the rotated line outside the central face can of course be removed by subsequent Z-invariant moves.

As a direct consequence of the YB equation, the order of the arguments may be permuted by

$$\begin{aligned} & F^{(i)}_{n}(z_1,\dots,z_{j-2},z_j,z_{j-1},z_{j+1},\dots,z_n)_{\varepsilon_1,\dots,\varepsilon_{j-2},\varepsilon_j,\varepsilon_{j-1},\varepsilon_{j+1},\dots,\varepsilon_n} \\ &= \sum_{\tau,\sigma} R(z_{j-1}/z_j)_{\sigma,\tau,\varepsilon_j,\varepsilon_{j-1}} F^{(i)}_{n}(z_1,\dots,z_{j-1},z_j,\dots,z_n)_{\varepsilon_1,\dots,\varepsilon_{j-2},\tau,\sigma,\varepsilon_{j+1},\dots,\varepsilon_n} \\ &\equiv \Big[R_{j-1,j}(z_{j-1}/z_j)F^{(i)}_{n}(z_1,\dots,z_{j-1},z_j,\dots,z_n)\Big]_{\varepsilon_1,\dots,\varepsilon_{j-1},\varepsilon_j,\dots,\varepsilon_n} \end{aligned} \tag{31}$$

This equation is graphically represented in fig. 12. The additional vertex with weight $R(z_{j-1}/z_j)$, that appears on the right-hand side, can be pushed up beyond the intersection with the horizontal line, and in fact all the way to the boundary, with no effect on the configuration sum. The contraction of the matrix $R(z_{j-1}/z_j)$ with the $j-1$-th and j-th edge variables variables ε_{j-1} and ε_j can be represented by an operator $R_{j-1,j}(z_{j-1}/z_j)$,

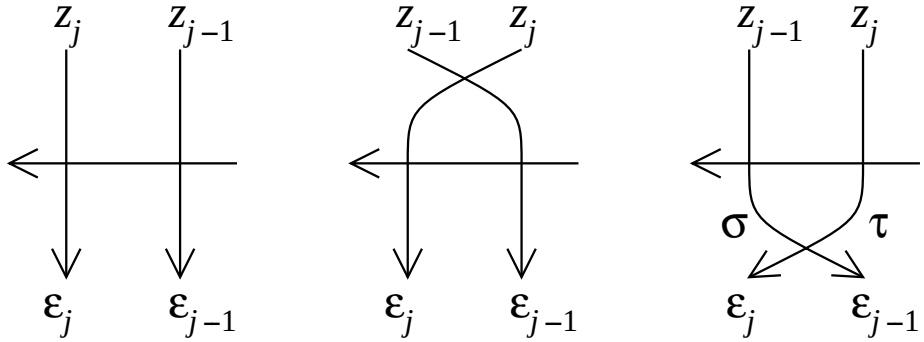

Figure 12. Interchange of two vertical lines with the corresponding spectral parameters, eq.(31). Starting at the rightmost figure, the diagonal vertex is first pushed beyond the horizontal line and subsequently away near the boundary.

and is in eq.(31) followed by the exchange of the $j-1$-th and j-th variable. In that notation

$$F_n^{(i+1)}(z_1,\ldots,z_j/x^2,\ldots,z_n) = R_{j+1,j}(x^2 z_{j+1}/z_j)\ldots R_{n,j}(x^2 z_n/z_j)$$
$$\times R_{1,j}(z_1/z_j)\ldots R_{j-1,j}(z_{j-1}/z_j) F_n^{(i)}(z_1,\ldots,z_j,\ldots,z_n) \tag{32}$$

is the consequence of (31) and (30). Eq. (31) is repeated to bring the j-th line all the way to the first position, then it is rotated via the lower edge of the central face to the last position by (30) and finally (31) is repeated again to move it from the last to the j-th position. The edge variable at the end of the line which is moved retains its j-th position throughout the process, while the spectral parameter z_j, and later $x^{-2}z_j$ moves through the argument list.

The equation (32) is a central ingredient to calculate the configuration sums for lattices with dislocations. It is convenient to make use of some other properties as well. Since the local Boltzmann weights are invariant for simultaneous inversion of all the states, so are the configuration sums.

$$F_n^{(i+1)}(z_1,\ldots,z_n)_{\varepsilon_1,\ldots,\varepsilon_n} = F_n^{(i)}(z_1,\ldots,z_n)_{-\varepsilon_1,\ldots,-\varepsilon_n} \tag{33}$$

Less obvious is the relation

$$F_n^{(i)}(z_1,\ldots,z_j,\ldots,z_n)_{\varepsilon_1,\ldots,\varepsilon_j,\ldots,\varepsilon_n} = (-)^{i+j}\varepsilon_j F_n^{(i)}(z_1,\ldots,-z_j,\ldots,z_n)_{\varepsilon_1,\ldots,\varepsilon_j,\ldots,\varepsilon_n}. \tag{34}$$

This can be seen as follows. The weights, a/κ and b/κ, of the ferromagnetic local states, are odd functions of of the spectral parameter, while c/κ and d/κ, of the antiferromagnetic states, are even. In the ground state all vertices are in the antiferromagnetic state. The sign of z_j affects only the vertices on the j-th line. The state $\varepsilon_j = (-)^{i+j}$ corresponds with the ground state configuration. Therefore an even number of the vertices in the j-th line is in a ferromagnetic state if $\varepsilon_j = (-)^{i+j}$, and an odd number if $\varepsilon_j = (-)^{i+j+1}$, hence (34).

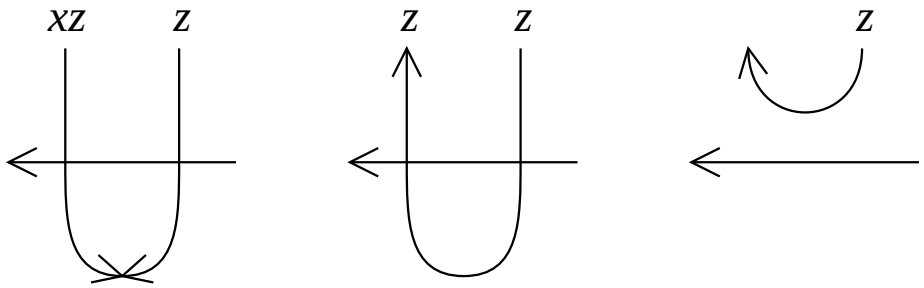

Figure 13. Normalization of the configuration sums, eq.(35), with the use of the inversion relation. The orientation of the line with spectral parameters xz is reversed by means of crossing symmetry, and the inversion relation is invoked to remove it together with the neighboring line with spectral parameter z.

The configuration sums can be normalized with the following equation

$$\sum_{\varepsilon} F^{(i)}_{n+2}(\ldots, z_{j-1}, xz, z, z_j, \ldots)_{\ldots,\varepsilon_{j-1},\varepsilon,-\varepsilon,\varepsilon_j,\ldots} = F^{(i)}_n(\ldots, z_{j-1}, z_j, \ldots)_{\ldots,\varepsilon_{j-1},\varepsilon_j,\ldots}. \tag{35}$$

The argument is given graphically in fig. 13. The line with spectral parameters xz can be turned into an upward oriented line with spectral parameter z, using crossing symmetry, fig. 8. The two lines with spectral parameter z, joint by the summation on ε, may be removed by the inversion relation (27).

A similar equation

$$F^{(i)}_{n+2}(\ldots, z_{j-1}, z, xz, z_j, \ldots)_{\ldots,\varepsilon_{j-1},\varepsilon,\varepsilon',\varepsilon_j,\ldots} = \delta_{\varepsilon,-\varepsilon'} F^{(i)}_n(\ldots, z_{j-1}, z_j, \ldots)_{\ldots,\varepsilon_{j-1},\varepsilon_j,\ldots} \tag{36}$$

is demonstrated as follows. Eq. (31) is applied to interchange the arguments z and xz, the special form (28) of $R(x)$ equates the two edge variables, and (35) removes two lines.

6. Staggered polarization

In this section we will utilize the equations (32) through (36) to calculate the spontaneous staggered polarization P of the eight-vertex model. The definition (*e.g.* [9]) is

$$P = F^{(1)}_{1,1}(z; z)_{+;+} - F^{(1)}_{1,1}(z; z)_{-;-} \tag{37}$$

provided the configuration sums can be interpreted as normalized probabilities. By means of (30) the function $F^{(1)}_{1,1}(z; z)_{\varepsilon;\varepsilon}$ is transformed into $F^{(2)}_2(xz, z)_{-\varepsilon,\varepsilon}$. To apply the functional equation (32) we consider $F^{(i)}_2(z_1, z_2)_{\varepsilon_1,\varepsilon_2}$. The spectral parameters of the lattice enter the configuration sums only as ratios of line spectral parameters. Therefore, if all line spectral parameters are multiplied with a common factor, the correlation functions do not change. Since we also know that F_2 depends only on two of the spectral parameters, it can only depend on the ratio. It is then convenient to denote $F^{(i)}_2(z_1, z_2)$ by $F^{(i)}_2(z_1/z_2)$. The difference equation (32) relates the four functions $F^{(0)}_2(z)_{+-}$, $F^{(0)}_2(z)_{-+}$, $F^{(1)}_2(z)_{+-}$

and $F_2^{(1)}(z)_{-+}$ among themselves. But (33) implies that $F_2^{(0)}(z)_{\varepsilon,-\varepsilon} = F_2^{(1)}(z)_{-\varepsilon,\varepsilon}$, which reduces the unknowns to two functions. The relation (32) for $j = 2$ is diagonalized by the introduction of

$$G^{\pm}(z) = F_2^{(1)}(z)_{+-} \pm F_2^{(0)}(z)_{+-} = F_2^{(1)}(z)_{+-} \pm F_2^{(1)}(z)_{-+}, \tag{38}$$

and now reads

$$G^{\pm}(x^2 z) = \frac{c_z \pm b_z}{\kappa_z} G^{\pm}(z). \tag{39}$$

Rather than solve both equations (39) separately, we note that $G^-(z) = G^+(-z)$ from (34), and need to work out only one case. It is convenient to have the equation in a factorized form, which can be done by the addition formula (15.4.28) of [1]. The form of $c_z + b_z$ can then be simplified using the identities $(-z;q)_\infty = (z^2;q^2)_\infty \,/\, (z;q)_\infty$ and $(z;q^2)_\infty \ (qz;q^2)_\infty = (z;q)_\infty$, with the result

$$c_z + b_z = q^{1/4} x^{-1} \frac{(xz;q)_\infty \ (qx^{-1}z^{-1};q)_\infty \ (x^2z^{-2};q)_\infty \ (qz^2x^{-2};q)_\infty \ (q^2;q^2)_\infty \ (q;q)_\infty^2}{(xz^{-1};q)_\infty \ (qx^{-1}z;q)_\infty}. \tag{40}$$

With these simplifications the equation for G^+ now reads

$$G^+(x^2 z) = \frac{f(z^{-1}) \ (xz;q)_\infty \ (qx_{-1}z^{-1};q)_\infty}{f(z) \ (xz^{-1};q)_\infty \ (qx^{-1}z;q)_\infty} G^+(z) \tag{41}$$

Thus we have a problem very similar to the equations (16) and (17) for the free energy. Again we assume that the solution is analytic and non-zero in the annulus $x - \epsilon < |z| < x^{-1} + \epsilon$. That implies that $\log(G^+)$ has a Laurent expansion convergent in that annulus. The terms of the expansion follow uniquely from (41) except the zeroth order term, which follows from the normalization (35). However since the functions involved are rather more complicated than for the calculation of κ_z, we will not construct the Laurent expansion. Instead we will construct a solution directly in product form, and make use of the above reasoning only to justify the uniqueness of the solution.

Define

$$\phi(z) = g(z) \ g(x^2 z^{-1}) \quad \text{with} \quad g(z) = \frac{(x^6 z^{-2};q,x^4,x^4)_\infty \ (qx^6 z^{-2};q,x^4,x^4)_\infty}{(x^4 z^2;q,x^4,x^4)_\infty \ (qz^2;q,x^4,x^4)_\infty}. \tag{42}$$

Then $g(z)$ satisfies the equation

$$g(x^2 z) = f(z^{-1}) g(z). \tag{43}$$

To work this out observe that the triple products implied in g are the same on either side, except for one of the boundaries. The boundary factors are double products which are taken care of by the $f(z^{-1})$. From (43) and the definition of ϕ (42) it is obvious that

$$\phi(x^2 z) = \frac{f(z^{-1})}{f(z)} \phi(z). \tag{44}$$

With this groundwork it is not difficult to verify that

$$G^+(z) \ = \ \frac{\phi(z)}{\phi(x)} \ \frac{(x^2;x^2)_\infty^2}{(q;q)_\infty^2} \ \frac{(qxz^{-1};q)_\infty \ (qx^{-1}z;q)_\infty}{(x^3z^{-1};x^2)_\infty \ (xz;x^2)_\infty} \tag{45}$$

solves (41) and satisfies the normalization $G^+(x) = 1$ from (35).

Finally, the spontaneous staggered polarization of the eight vertex model now follows from (37), (30), (38) and (34).

$$P = G^+(-x) = \frac{(-q;q)_\infty^2 \ (x^2;x^2)_\infty^2}{(q;q)_\infty^2 \ (-x^2;x^2)_\infty^2} \tag{46}$$

Even though this formula was derived for the first time [4] by the method presented here, it was conjectured before by Baxter and Kelland [10], on the basis of the result for the six-vertex model [9], $q = 0$ and for the decoupling limit $x = q^{1/4}$.

Of course the staggered polarization of the uniform eight-vertex model is only a special case of the quantity we calculated. The result for G^+ also determines the expectation value of the edge variable at a lattice defect where the spectral parameter of one of the lines changes its value, or where two half lines may run off in arbitrary directions. One should be aware, however, that in that case the argument, that the partition sum of the lattice with normalized vertex weights is unity, is no longer valid, as the broken lines cannot be moved arbitrarily by Z-invariant moves. In effect the dislocation inserts a boundary in the middle of the lattice, and the argument leading to the locality of thermodynamic function does not apply at or near the boundary. The functions $F_2^{(i)}$ should then be viewed only as unnormalized configuration sums. Of course probabilities can still be recovered by dividing by appropriate sums of configuration sums. For example the polarization of a vertical edge variable on a line of which the spectral parameter is z_1 above and z_2 below that edge is given by

$$P = \frac{G^+(-xz_2/z_1)}{G^+(xz_2/z_1)} \tag{47}$$

7. conclusion

In these lectures we have discussed the Yang-Baxter equations and some of its consequences for lattice models in two dimensions, explicitly for the eight-vertex model. The inversion relation method, presented in this school by Baxter [6] thirteen years ago, is an elegant method and a simple tool to calculate the free energy. The method depends essentially on an analyticity assumption but is otherwise exact.

Difference equations for correlation functions of the eight vertex model are shown to be based almost exclusively on the Yang-Baxter equation and the inversion relation. The simplest case results in a calculation of the spontaneous staggered polarization. The method in principle allows the calculation of general correlation functions. As the system of difference equations gets rapidly more complicated with the distance between and the number of correlated variables, it remains to be seen how practical the method is for other cases. Even if the results of the method will in the end be limited to only small distance correlations, it is quite an intriguing fact that these results can be obtained with such simple methods.

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