Loop models

Bernard Nienhuis
b.nienhuis@uva.nl

Preface for the summerschool LDQS 2014

These notes have been prepared for another occasion, and in this school I will use only section 3, and not even all of it. But for those interested, section 2 gives a very compact description of how the Renormalization Group (RG) in condensed matter systems has shaped our thinking about critical systems. It is written with classical systems in mind, but is equally applicable to critical quantum systems. And section 4 gives an introduction how in two dimensions RG calculations have predicted/confirmed exact critical exponents for many systems.

1 Historical perspective

Loop models have come up in the theory of phase transitions and (other) phenomena involving spatial scale invariance. One may say that the best understanding of phase transitions has come from a succession of theories that evolved in the 20-th century. A phase transition can be defined as a locus in parameter space across which matter has very different properties. This implies that at a phase transition thermodynamic functions are singular, i.e. non-analytic, as functions of the parameters. When the two different states of matter can co-exist, the transition is called first order, and otherwise it is continuous. A continuous phase transition is also called a critical point. At and near critical points physical observables are related via power laws. The powers appearing in these relations, the critical exponents, became the focus of attention. The first breakthrough was mean Field (MF) theory, which was capable of predicting phase transition starting from a microscopic description. In this approximate approach, the critical exponents can take only few values typically independent of the dimensions of the system. Over time it was observed that, indeed critical exponents are highly universal, but do depend on the dimension of the system, and on the symmetry involved in the phase transition. The exponents are insensitive to the details of the interaction. MF remained the dominant theory until Renormalization Group (RG) took over around 1970. Renormalization theories implement explicitly the concept of spatial scale invariance. Two approaches were developed. One, called real space RG[1, 2], where a scale transformation is expressed in the variables labeled by spatial coordinates. The other, momentum space RG[3] used the variables labeled by coordinates in the Brioullin zone. Real space RG was mainly done in two dimensions (2D), as this is more accessible to explicit numerical calculation. Momentum space focussed on an expansion in powers of ε, where the dimension of space is $4-\varepsilon$.

It was discovered that many two-dimensional models popular for the study of phase transitions, could be formulated in terms of electric charges and magnetic monopoles interacting via the the 2D version of electromagnetism.[4, 5] Combined with RG, this *Coulomb Gas* (CG) method proved to be an effective way to calculate universal quantities. In this approach loop models form the vocabulary of an intermediate language between, on one hand lattice models with local interactions, and Coulomb Gases on the other. Loop models are ensembles of paths on the lattice with a statistical weight determined by the configuration of the steps in the paths and the total number of closed paths. Because the paths may not terminate in the interior of the lattice, they either form closed loops or run between two points of the boundary.

A natural extension of scale invariance is its extension from a global to a local symmetry, invariance under conformal transformations of space. The notion that conformal invariance is

both plausible and powerful led to the study of Conformal Field Theory (CFT) as the scaling limit of critical systems. It proved very effective in predicting many properties of the scaling limit. Finally around 2000 the Stochastic Löwner evolutions (SLE) allowed the mathematicians to prove what physicist had been so busy calculating all these years. While CFT, and (for rigorous arguments) SLE are currently more powerful than CG, the latter approach has some clear advantages. In the CG approach, a lattice model and its operators are translated directly into a gas of interacting particles with specific properties, while in SLE and CFT the connections is indirect and based on the correspondence of emerging properties.

Simultaneous with this development, a sequence of specific lattice models was solved exactly. The solution to the 2D Ising model had already shed doubt on the validity of MF theory in 1944[6]. Much later followed the 6-vertex model[7], and the 8-vertex model[8], the hard square[9] and hard hexagon[10] model. In particular the formulation of the Yang Baxter equation (YBE)[11] led to an explosion of solvable models. These solutions were instrumental in the development of the theory of phase transitions, and served as anchor points for approximate or speculative approaches. It is noteworthy that, unlike MF theory, all predictions of RG theory are consistent with the existing exact solutions.

2 Brief summary of renormalization theory

The ideas that led to the application of renormalization (usually designated Renormalization Group, or RG), to condensed matter systems can be formulated as follows. A many body system typically has a number of relevant spatial scales. The microscopic scale is e.g. the size of the molecules or the lattice constant (in case the lattice is considered as given). Emerging length scales are correlation lengths, screening lengths and the like. In generic systems these various length scales are of the same order of magnitude. However, when a continuous phase transition (or critical point) is approached the correlation lengths grow, and can become macroscopic. This implies that macroscopic properties involving space have a characteristic scale. However, when the critical point is reached, this spatial scale diverges. As a consequence the characteristic scale disappears, and the observables become scale invariant (that is on scales sufficiently removed from the microscopic scales). If this train of thought is taken seriously then it is plausible that one can formulate a scale transformation for which the critical system is invariant and the generic system covariant. The mere existence of such transformation has many important consequences that can be verified. If the transformation can be calculated or approximated even more can be deduced.

To make this explicit, consider a large but finite statistical system with N degrees of freedom s, associated with the sites of a lattice. It has a hamiltonian $H_N(g,s)$, where g stands for a number of coupling constants. The partition sum (or integral) is given by

$$Z_N(g) = \sum_s \mathrm{e}^{-H_N(g,s)} \tag{1}$$

(the factor kT is absorbed in the coupling constants). To formulate a renormalization transformation (RT) we assume that we can course-grain the variables to a smaller number, say N'. This means we split the degrees of freedom s in a two sets s' and σ that describes the long and short wavelength fluctuations respectively. Then the summation of the Boltzmann weight over the short-wavelength variables is performed, so that the sum over the remaining variables would result in the original partition sum. The summand after the partial summation is written as an effective Boltzmann weight again, so that

$$Z_N(g) = \sum_s \mathrm{e}^{-H_N(g,s)} = \sum_{s'}\sum_{\sigma} \mathrm{e}^{-H_N[g,s(s',\sigma)]} = \sum_{s'} \mathrm{e}^{-H_{N'}(g',s')} = Z_{N'}(g'), \tag{2}$$

where s represent the original variables, and s' and σ only the long and short wavelength fluctuations respectively. The notation reflects the hypothesis that the partially summed Boltzmann weight, can be written as the exponent of an effective Hamiltonian of the smaller system of the same form as the original Hamiltonian but with altered values of the coupling constants. We further assume that the function $g'(g)$ is analytic or at least differentiable even at a phase transition. There is, at this point, no need to assume that one knows how calculate $g'(g)$, only that in principle it exists.

Let $N = \ell^d N'$, with d the dimension of space, so that the linear change of scale is ℓ. In the limit that N is very large the free energy is proportional to N, so that the free energy per lattice site satisfies

$$f[g'(g)] = \ell^d f[g] \tag{3}$$

Going back to the correlation lengths, we introduce the symbol ξ for one of the correlation lengths, in units of the lattice constant. This must satisfy

$$\xi[g] = \ell \xi[g'(g)] \ . \tag{4}$$

Since criticality corresponds with the divergence of ξ, equation (4) shows that the parameters g being a critical point, implies that also $g'(g)$ is a critical point. In other words critical manifolds are invariant under the map $g'(g)$. We may then assume that a critical manifold contains fixed points.

$$g'(g^*) = g^* \ , \tag{5}$$

which are going to play a leading role as indicators of phase transitions.

Exercise 2.1 *What possible values can $\xi[g^*]$ have if g^* is defined by (5)*

Exercise 2.2 *Equation (3) ignores the fact that integration (or summation) of a part of the degrees of freedom, typically leads to an additional term in the Hamiltonian, independent of the remaining s'. Taking this into account, leads to an inhomogeneous version of (3)*

$$f[g'(g)] = \ell^d \, f[g] + h[g], \tag{6}$$

In which both $g'(g)$ and $h[g]$ are analytic functions. Show that one can write the free energy as the sum of an analytic part and a singular part, $f[g] = f_{\rm a}[g] + f_{\rm s}[g]$, such that $f_{\rm s}$ satisfies (3).

If the microscopic scale is not given by a lattice, but by a UV integration cut-off, it is convenient to take ℓ as a variable, and analyse

$$\beta(g) \equiv \lim_{\ell \to 1} \frac{\partial g'(g)}{\partial \ell} \tag{7}$$

called the β-function. In this case the fixed point equation is $\beta(g^*) = 0$.

What can we say about the behavior of the system in the vicinity of a fixed point? Sufficiently near g^* we can linearize the transformation, and diagonalize the derivative matrix $\partial g'(g)/\partial g$. The principle variations of g near g^* we call u_j, and we can write the free energy and correlation length in terms of these u_j. The significance of a specific u_j depends on the corresponding eigenvalue. If an eigenvalue (assumed to be real) is larger than unity, the corresponding u_j grows, and if it is less than 1, u_j decreases.

Since the eigenvalues depend on ℓ, it is useful to make that dependence explicit. Since successive RT's with rescaling factors ℓ and ℓ' correspond to a rescaling with factor $\ell \, \ell'$, and also the eigenvalues multiply, we write these eigenvalues in the form ℓ^y. To illustrate the distinction of eigenvalues larger or less than one, or equivalently positive and negative exponent y, we first discuss a case with two coupling constants, and $y_2 < 0 < y_1$. There is a non-linear coordinate

transformation $u(g)$ from g to u, with $u' = u(g')$, so that in terms of u the RG transformation has the simple form $u_i' = \ell^{y_i} u_i$. Considering the 'flow lines' (i.e. the curves in the parameter space traced out by progressive renormalization), in figure 1, shows the line $u_1 = 0$ is a watershed between different limits of the flows. This implies that the physical behavior on either side of the curve $u_1 = 0$ is very different. If e.g. the effective variables at a very coarse scale are strongly coupled, the original variables will have long ranged correlations, and if the effective variables are weakly coupled, the original variables have only short range correlations. Clearly the locus $u_1 = 0$ is that of a phase transition.

Exercise 2.3 *Argue why the u_1-axis is generally not a phase transition. Try to think of an extension of the RG flow lines outside of the borders of figure 1, which would form an exception.*

When we write the free energy and correlation length as functions of u, the functions, $\xi(u_1, u_2)$ and $f(u_1, u_2)$, satisfy

$$\begin{aligned} \xi(u_1, u_2) &= \ell \xi \left(u_1 \ell^{y_1}, u_2 \ell^{y_2}\right) \\ f(u_1, u_2) &= \ell^{-d} f \left(u_1 \ell^{y_1}, u_2 \ell^{y_2}\right) \end{aligned} \tag{8}$$

the solutions of these equation are given by

$$\begin{aligned} \xi(u_1, u_2) &= |u_1|^{-1/y_1} \; X_\pm \left(u_2 \, |u_1|^{-y_2/y_1}\right) \\ f(u_1, u_2) &= |u_1|^{d/y_1} \; S_\pm \left(u_2 \, |u_1|^{-y_2/y_1}\right) \; , \end{aligned} \tag{9}$$

where X and S are unknown functions, and their suffix is the sign of u_1. The functions X and S are called **scaling functions**, noting that their argument does not change under rescaling.

Exercise 2.4 *Show that indeed (9) are solutions of (8) and that all solutions are of this form if ℓ can be chosen arbitrarily.*

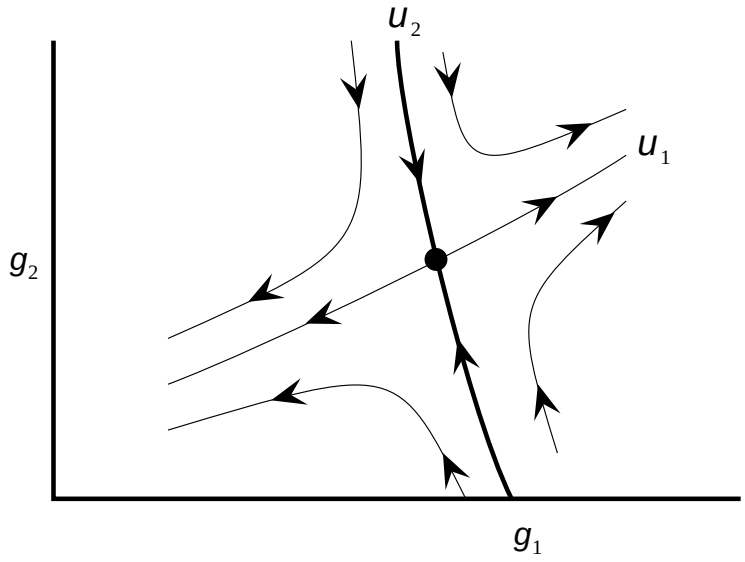


Figure 1: The renormalization flows near a fixed point with one relevant and one irrelevant variable.

In general the locus $u_2 = 0$ is not a phase transition. This implies that the thermodynamic functions are *analytic* at $u_2 = 0$ and the functions X and S can be expanded in powers of their argument.

$$\xi(u_1, u_2) = \sum_{k=0}^{\infty} X_{k,\pm}\, u_2^k\, |u_1|^{(-1-k\,y_2)/y_1}$$

$$f(u_1, u_2) = \sum_{k=0}^{\infty} S_{k,\pm}\, u_2^k\, |u_1|^{(d-k\,y_2)/y_1} \tag{10}$$

Even though we have only assumed the existence of an RG transformation, and not its precise form, this result for the thermodynamic functions is very informative. We see here that the singular behavior of the free energy and the correlation length are a sum of power-law terms, with increasing exponent. The non-integer powers concern the u_1 dependence. The leading term is determined by y_1. None of the exponents depends on the value of u_2. Because the leading critical behavior is not affected by u_2, this parameter is called **irrelevant**. In contrast, the parameters with positive exponent y are called **relevant**. When $y = 0$ the corresponding parameter (or operator) is called marginal. Relevant (irrelevant) parameters increase (decrease) under repeated RG transformation.

Exercise 2.5 *Argue that a marginal operator can be marginally relevant, marginally irrelevant, that this behavior will generally be different on either side of the fixed point, but need not be, and finally that one can also have strictly marginal operators.*

The amplitudes of the non-dominant terms depend explicitly on the value of the irrelevant parameter. When the nonlinear terms of $g'(g)$ (and thus of $u(g)$) are taken into account, also the coefficient of the leading term will depend on u_2. Finally, note that the exponents of the free energy and of the correlation length are not independent.

Now consider a case with two relevant variables, $y_1 > y_2 > 0$, omitting possible irrelevant variables. To make contact with some real life case, think of u_1 as the magnetic field, and of u_2 as the temperature variation in the neighborhood of the critical temperature. Equation (9) for the free energy can be used to calculate the various thermodynamic functions: magnetization M and susceptibility χ as first and second derivative of f w.r.t. the field h, the heat capacity C as second derivative of f w.r.t. the temperature T. The well-known critical exponents (at zero field: $C \propto |T - T_c|^{-\alpha}$, $M \propto (T_c - T)^{\beta}$, $\chi \propto |T - T_c|^{-\gamma}$, $\xi \propto |T - T_c|^{-\nu}$, and at $T = T_c$: $h \propto M^{\delta}$) can now readily be calculated. We get:

$$\alpha = 2 - \frac{d}{y_2}, \quad \beta = \frac{d - y_1}{y_2}, \quad \gamma = \frac{2y_1 - d}{y_2}, \quad \delta = \frac{y_1}{d - y_1}, \quad \nu = \frac{1}{y_2} \tag{11}$$

Exercise 2.6 *Verify these values of the critical exponents.*

Evidently these exponents are not independent but can expressed in the two eigenvalues y_1 and y_2. The generalization to more than two parameters, some relevant and some irrelevant does not introduce essential difficulties.

Also the overall scaling of critical correlation functions can be calculated within the RG framework. Let the (extensive) operator conjugate to one of the parameters u_j be Q_j. Now consider the fluctuation of Q_j in a finite system with linear size L at a point in parameter space which is separated from the fixed point only in the u_j-direction.

$$G(u_j, L) \equiv \langle Q_j Q_j \rangle - \langle Q_j \rangle^2 = L^d \frac{\partial^2 f_L(u_j)}{\partial u_j^2} \tag{12}$$

Analogous to (8) this scales as

$$G(u_j, L) = \ell^{2y_j}\ G(\ell^{y_j}\ u_j, L/\ell) \tag{13}$$

so that

$$G(0, L) \propto L^{2y_j} \tag{14}$$

Let Q_j be written as a sum (or integral) over local operators:

$$Q_j = \int_{L^d} \mathrm{d}^d r\ q_j(r) \tag{15}$$

then G can be written as an integral over the two-point function of $q_j(r)$:

$$G(0, L) = \int_{L^d} \mathrm{d}^d r \int_{L^d} \mathrm{d}^d r'\ \left\langle\, q_j(r)\ q_j(r')\,\right\rangle\ -\ \left\langle\, q_j(r)\,\right\rangle\ \left\langle\, q_j(r')\,\right\rangle \tag{16}$$

Since the total integral behaves as a power of L (14), the integrand is expected to behave as a power of the distance

$$\left\langle\, q_j(0)\ q_j(r)\,\right\rangle\ -\ \left\langle\, q_j(0)\,\right\rangle\ \left\langle\, q_j(r)\,\right\rangle\ \propto\ |r|^{2(y_j - d)}\ . \tag{17}$$

It is convenient to introduce $x_j = d - y_j$, because these are the exponents that appear in correlation functions. We thus see that both relevant and irrelevant operators have a power law decay for their two-point function. If the decay is faster than $|r|^{-2d}$, the operator is irrelevant, and if the decay is slower, the operator is relevant.

2.1 Summary

The existence of an analytic RT is a strong and unproven assumption. It has among others the following consequences:

1. **Relevant** and **irrelevant** parameters ($y \gtrless 0$) increase and decrease under (repeated) renormalization.
2. Critical exponents do not depend on irrelevant parameters, but can only depend on invariants of the RT, e.g. symmetry and spatial dimension. In particular, they do not depend on the lattice (unless the symmetry of the lattice is an essential ingredient in the transition). This claim is known as **universality**.
3. Two-point functions decay with a power of the distance equal to $2x_j = 2(d - y_j)$, where y_j is the exponent of the coupling parameter conjugate to the operators correlated.
4. All correlation lengths diverge with the same exponent, when a critical point is approached.
5. Critical exponents are the same on either side of the transition, but the corresponding amplitudes may differ.
6. Critical amplitudes of dominant singularities and of corrections to scaling may depend on the irrelevant parameters.
7. The ratio of critical amplitudes of the same observable on either side of the transition is universal.
8. Scaling functions, i.e. the singular dependence of a thermodynamic quantity of two or more (relevant) thermodynamic parameters is universal.

9. The exponents for the various thermodynamic quantities are not independent, but satisfy a number of scaling relations.

10. When integer linear combinations of the exponents y add up to d, corresponding integer power laws are modified with a logarithm. This is not derived in the text above, but left as an exercise.

11. RG theory admits the existence of a line of fixed points, along which critical exponents may vary.

12. At the point where an irrelevant parameter turns relevant, there are logarithmic corrections to the power law behavior.

Exercise 2.7 *Argue that universality is a natural consequence of the existence of a RT.*

Exercise 2.8 *Verify that all correlation lengths diverge with the same exponent as a critical point is approached (item number 4)*

Exercise 2.9 *Verify that amplitude ratios are universal (item 7).*

Exercise 2.10 *As an example of universal scaling functions (item 8) one may think of the magnetization M of a ferromagnet near the Curie point ($T = T_c + \varepsilon$) in a small magnetic field h. $M(\varepsilon, h)$. This function, which has a discontinuity at $\varepsilon < 0$ and $h = 0$, in the limit $h \to 0$, and $\varepsilon \to 0$, and up to an overall scale in h, ε and M is universal. Verify this by formally calculating M from the RG equations in the presence of h, ε and one irrelevant variable.*

Exercise 2.11 *As an example of logarithmic corrections (item 10) consider a one-parameter RT of which the scaling exponent y=$d/2$. Show that an attempt to find the analytic solution to the scaling equation (6) fails, and can be salvaged by a logarithmic factor. In this case there is no analytic solution to (6), but instead the solutions to the (3) are analytic.*

In all of the many cases where exact solutions of model systems or accurate experimental or numerical data are available, these consequences of the RG theory are corroborated. For physics this is the ultimate test of a theory. And for this reason the RG theory of phase transitions has the same status as any of the established theories in physics.

3 Loop models

Loop models are (lattice) gases of closed paths or loops. In the cases discussed here, the loops do not intersect themselves or each other, but other cases have been considered. Paths may be infinite or terminate on the boundary of the system, in which case they are not strictly loops. The models came up in the theory of phase transitions, but have been used also in other condensed matter contexts.

The possible configurations of loop segments around a single vertex are shown in figure 2. We consider here in particular the square and hexagonal lattice, but the generalization to other lattices possibly in higher dimension is immediate.

The different local configurations have different weights, which may depend on orientation, and each closed loop has weight n. The partition sum is thus given by

$$Z_{\text{loop}} = \sum_{\text{cfg.}} n^N \prod_v W(v) \tag{18}$$

The sum is over all loop configurations, N is the number of loops, so that each loop has weight n. The product is over the vertices v, where $W_{\text{local}}(v)$ is the weight of the configuration at vertex v, and depends only on which of the elements of figure 2 is present, and possibly its orientation.

For the loop model on the hexagonal lattice with the full symmetry of the lattice, the partition sum simplifies to

$$Z_{\text{hex}} = \sum_{\text{cfg.}} n^N \, x^L \,, \tag{19}$$

where L is the total length of all loops together. Here the weight of the two local vertices in figure 2 is chosen 1 and x respectively.

The n=0 loop model denotes not just the $n \to 0$ limit, which is trivial, but the derivative of the partition sum w.r.t. n which generates the configurations of a single loop. Higher derivatives describe a fixed number of such loops. Likewise the correlation function of two operator insertions at which a path terminates, gives the partition sum of an open chain.

Exercise 3.1 *Locality of a statistical model can be defined as the property that the change of the Boltzmann weight due to a local change of the configuration can be determined by inspection in the neighborhood of the locus of the change. Verify that the loop model for generic n is not local in this sense. For what value(s) of n should one make an exception?*

Exercise 3.2 *Show that the hexagonal version can be recovered from the square by allowing the weights to be anisotropic, and demanding that the weight on the square factorizes into two triangles.*

Similar models on the triangular lattice or more general semi-regular or even random lattices can be formulated analogously. Also, intersections can be introduced, or additional variables decorating the loops.[12, 13] These also have various applications in physical systems, and have been studied for this reason. Here we consider only the simplest versions.

In any particular model, the configurations shown in figure 2 need not all have non-zero weight. A well studied case is the Completely Packed Loop (CPL) model on the square lattice, in which the first three vertex configuration of figure 2 have zero weight. In this model every edge is visited by a loop, and every vertex is even visited twice. In the regime that the loops have a weight $-2 \leq n \leq 2$, the CPL model is critical, meaning that the correlation functions decay algebraically with distance. If only the first and the last are omitted, every *vertex* is visited once: this is called the Fully Packed loop (FPL) model. The CPL model can be defined also on the triangular lattice, but not on the hexagonal lattice. The FPL model exists for all three.

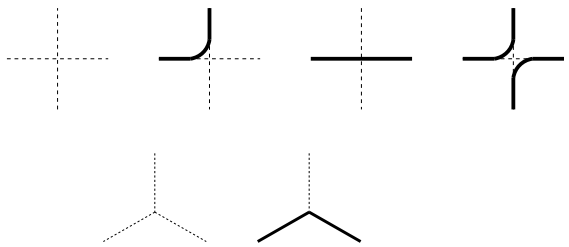


Figure 2: The local configurations of paths around a single vertex, for the square lattice (top) and hexagonal lattice (bottom). The vertex types are shown modulo the symmetry group of the lattice.

When all vertices have non-zero weight (on the square or hexagonal lattice), the density of loops can be controlled by the weight of each vertex configuration. The model has a non-critical phase where the vertex weights are dominated by the empty vertex. When the weight of the other vertices is increased, the model undergoes a phase transition into a critical phase with loops of great length. In this phase the model is called the dense loop model, and at the phase transition it is called the dilute loop model. The dense loop model is distinct from the FPL and CPL models in the fact that the density is fluctuating.

The critical behavior in the dense phase, and at the transition between the dense and the non-critical phase, is characterized by exponents that depend continuously on n, but do not depend on the precise value of the weights associated with the local vertex configurations. The next section will show that many models with local degrees of freedom and local interactions, can be mapped onto loop models with a specific value of n. This is used as a tool to identify the universality class of the transitions, because that depends on n alone.

3.1 ADE models

Loop models can be used to represent lattice models with local variables and local interaction. One large class of such models are known as ADE models. In these models the faces of the lattice assume discrete values, represented by the nodes on a graph $\mathcal{G}$. Adjacent faces can either take the same value, or different values connected by a link in $\mathcal{G}$. The name ADE comes from the classification of graphs; Classes called A, D, and E play a special role (see figure 3). The graph $\mathcal{G}$induces a adjacency matrix A with elements $A_{i,j}$ equal to 1 if the nodes i and j are connected and 0 otherwise. Now we are ready to define the ADE model. [14] We start with the configurations of a loop model. The loops are interpreted as domain walls between different domain. In each domain the faces all take the same value, while states on either side of a domain wall must be adjacent on $\mathcal{G}$. The weight of the ADE model is completely local, and written as a product over the vertices:

$$W_{\mathrm{ADE}} = \prod_{\text{vert } v} W(v) \prod_{\text{turns}} A_{i,j} \left(\frac{S_j}{S_i}\right)^{\gamma_{\mathrm{b}}/2\pi} . \tag{20}$$

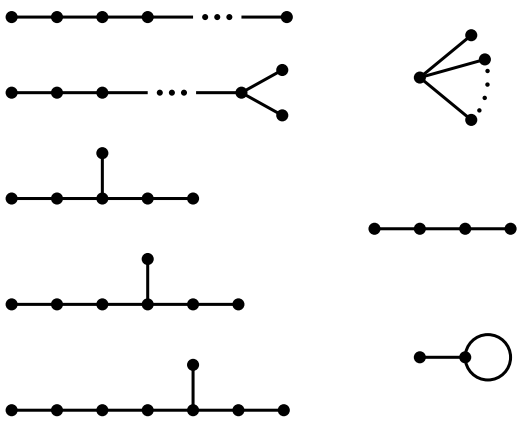


Figure 3: The adjacency diagrams. In the left column the top figure is the A_m diagram, the second line the D_m and the last three lines E_6, E_7 and E_8. The suffix refers to the number of nodes. In the right column respectively the adjacency diagram of the Potts model, the diagram A_4 and its folded version for the hard hexagon and hard square model.

There is a factor $W(v)$ for each vertex depending only on the local configuration of domain walls, equal to the weight of the corresponding vertex type of the loop model. And there is a factor for each turn of the domain wall, which depends on the states of the faces j and i on either side of the domain wall, and on the angle over which the domain wall bends at the vertex. The bending angles $\gamma_{\rm b}$ in (20) are counted positive when the state j is on the inside of the bend (and i on the outside). We choose S_j to be the j-th element of an eigenvector of A and let n be the corresponding eigenvalue: $\sum_j A_{i,j} S_j = n S_i$. If we choose S to be the Perron-Frobenius eigenvector, with only positive elements, then the local weights are real and positive. Since the bending angle summed over a closed loop is $\pm 2\pi$, the corresponding weight of a closed domain wall (apart from the weight coming from the factors $W(v)$) is

$$W_{\rm loop} = A_{i,j} \frac{S_j}{S_i} \,, \tag{21}$$

where j is the state on the inside of the domain wall and i outside. When for a fixed configuration of domain walls the sum over compatible state configurations is performed, each closed domain wall thus contributes a factor equal to the eigenvalue of A for eigenvector S, i.e. n. To see this one has to start summing over the state of the innermost domains, keeping the value in the other domains fixed. This results only in the factor n. Subsequently one recursively sums over the state of the domains surrounding the ones just summed out. This works perfect when all domain walls are contractible, e.g. in a bounded region in which the boundary faces are all in the same state, and results in the partition sum (18).

With other boundary conditions the paths may terminate on the boundary, and these paths may have a weight different from n, depending on the specific boundary condition. Also on a torus there may be non-contractible loops, which wind around the torus. These do not have the weight $A_{j,k}\ S_k/S_j$, but simply $A_{j,k}$. The domains which are separated by these winding loops form a closed chain. Once all domains surrounded by contractible walls are summed over, the only sum that remains is over the non-contractible domains. This sum is the trace of a power of the matrix A. As a result the ADE partition sum *on the torus* therefore reads,

$$Z_{\rm ADE,T} = \sum_j \sum_{\rm cfg} n^{N_{\rm c}}\, \Lambda_j^{N_{\rm w}} \prod_{\rm vert\ } W(v) \tag{22}$$

where the first sum is over the eigenvalues Λ_j of A, the second over all loop configurations, and $N_{\rm c}$ is the number of contractible loops, and $N_{\rm w}$ of winding loops. For those familiar with CFT: each term in the sum on j corresponds to a character in the modular partition sum.[16]

Exercise 3.3 *To follow this construction in detail, it is useful to represent each loop configuration by a graph in which the nodes are the domains and the links are the domain walls between them. Note that in a region with genus zero such graph is a tree, while on the torus the graph may contain one closed cycle.*

Other eigenvalues of A than only the largest one, can also be used as a possible value for n. In this case the ADE model may have complex weights, but this is not necessarily the case. The model will correspond to a CFT with negative conformal weights, which can be a blessing or a curse. It may be avoided by some selection rule that excludes the negative weight sectors, corresponding to terms in the sum on j in (22) in which $\Lambda_j > n$.

The equivalence between loop models on one hand and discrete lattice models with local weights on the other, was first shown by Pasquier[14] for the CPL model, and later generalized to general loop models [17]. We will refer to the translation as ADE construction.

3.1.1 ABF models

The ADE models with adjacency diagram A_m have been studied by Andrews, Baxter and Forrester, and are called ABF models or restricted solid-on-solid model. It can be viewed as a spin-(m-1)/2 Ising model. The critical point obtained from the ADE construction based on the CPL model, is the Ising critical point for m=3, and for larger values of m the models give a sequence of multicritical points, each one selecting the point in parameter space where the transition corresponding to the previous model turns first-order.

Transitions in the same universality class are found for loop models in the dense phase instead of the CPL model. In the dilute loop models the same sequence of universality classes is found, with m one less.

Exercise 3.4 *Calculate the loop weights n for the ABF models corresponding to diagram A_m.*

3.1.2 The height models

The A_m models in the limit $m \to \infty$ will play an important role in these lectures, since they are the basis for the Coulomb Gas (CG) theory. The position along the diagram is sometimes referred to as height, and the model as height model. The adjacency diagram has a continuous spectrum of eigenvalues $-2 \leq \Lambda \leq 2$, which makes this models very flexible. Also they have running wave modes as eigenstates, so that the elements of the eigenvector are all the same in magnitude, only different in phase. We will use them mainly as a calculational tool: they admit any possible loop weight in the interval [-2,2], and the operators in other models can usually be translated in the A_∞ language.

The height model with n=2 is used to describe the behavior of a crystal surface. Each height representing a stack of atoms. The non-critical phase is called smooth, referring to the fact that all the heights globaly concentrate around one given value (symmetry breaking of the $\mathbb{Z}$ symmetry). The regions of other heights form small islands or (lakes) and occasionally islands within islands.

The dense phase corresponds to the rough phase of the crystal surface, because here the heights fluctuate wildly, the local averages of heights take continous values, and the system (on a large scale) no longer has preference for integer values. The transition between the smooth and the rough phase is called the roughening transition. In real crystals (and in sufficiently rich models) the roughening temperature depends on the direction in which the height is measured. The roughening temperature is lowest in the directions of high symmetry. As a result a crystal in equilibrium with a melt, a vapor or a solution, has smooth facets separated by rounded (rough) edges and corners. Experimental findings are in good agreement with the theory. A more open question is the shape of a crystal in various conditions of growth outside of equilibrium.

If the CPL model is used as the basis of the A_∞ model, the weights are real positive irrespective of the value of n, in spite of the fact that the eigenvector is complex. This is because in the local loop configurations two bends always come together. Each bend carries a complex weight, say z, if the height on the inside of the bend is greater, and z^*, when it is less than that on the outside. When the four heights around a single vertex have an overall gradient, the factor z for the one bend, is multiplied with z^* for the other. When the four heights form a saddle configuration, the weight z is multplied with itself, but this height configuration is compatible with both vertex types, and if one has weight z^2, the other has weight z^{*2}. Thus the sum is real and in fact non-negative.

3.1.3 The Potts model

Since symmetry is an important determinant of a universality class, it is useful to construct adjacency diagrams with special symmetries. An extensively investigated model is the Potts

model[18] with S_q, the permutation group of q elements, as its symmetry. Its critical point can be found with an ADE construction. The appropriate state diagram is shown in the second column of figure 3. It has one central node, and q legs. We refer to the q nodes as 1,2,$\cdots, q$, and to the neutral or central node as 0.

Exercise 3.5 *Verify that the largest eigenvalue is $\sqrt{q}$ and the corresponding eigenvector has the form $(\sqrt{q}, 1, 1, \ldots)$.*

It follows that the loop weight of the Potts model is $\sqrt{q}$. In the (homogeneous and isotropic) CPL model, since the square lattice is bipartite, one sublattice is completely frozen in the neutral state, while the equivalent q states, called spins, live on the other sublattice. One may ignore the frozen sublattice and deal with the effective interactions on the dynamic sublattice only. The Perron-Frobenius eigenvector has weight 1 for each of the q equivalent nodes, and $\sqrt{q}$ on the neutral node (up to normalization).

Exercise 3.6 *Verify that in each square of four faces of the lattice, the weight of two unequal states across the diagonal of a square is 1, while the weight of two equal states is 1+$\sqrt{q}$.*

Another type Potts model is found if instead of the largest eigenvalue $\sqrt{q}$ we choose its conjugate $-\sqrt{q}$. This critical point is in the antiferromagnetic regime, i.e. the spins are preferentially different from their neighbors.

In principle we can use any loop model in this construction, but then all q+1 states can sit anywhere in the lattice. Such models are known as *dilute* Potts models, referring to the neutral state as a vacancy or an unoccupied site. Dilute Potts models have a richer phase diagram than simple Potts models, as the transition for any value of q can be driven first-order, with a tricritical point separating the continuous and discontinuous branches of the transition. All this is well represented in the loop model associated to the dilute Potts model by the ADE construction.

3.1.4 Percolation, UST, SAW and θ-SAW

Percolation is a description of transport behavior in a random mixture of permeable and non-permeable materials (or conducting and insulating materials). One prototypical model is site percolation on the triangular lattice. It is a perfect model for a mixture spherical beads of copper and glass on a tray, (settled in a closed-packed array). Only when more than half of the beads is copper, a large array will conduct. This model, at the transition, is well described by the hexagonal loop model with n=1 and x=1 (the loops being the boundary between the conducting and non-conducting beads). Another prototypical model is bond percolation on the square lattice. In bond percolation neighboring sites are connected (by a conductor) with probability p, independently of each other. Again precisely at p=1/2 the model undergoes a transition between a conductor and an insulator. This model at the transition is described by the CPL loop model with n=1. This fact makes it evident that percolation is the $q \to 1$ limit of the Potts model.

Uniform spanning trees (UST) are the set of all tree-like subgraphs of a lattice, chosen with uniform weight. The CPL loop model with n=0 is a description of this ensemble. The UST are *the* (or rather *a*) $q \to 0$ limit of the critical Potts model[19].

Exercise 3.7 *Convince yourself of the relations between the two percolation models and the UST model on one hand, and the corresponding loop models on the other.*

Self avoiding walks (SAW) are used to describe linear polymers in solution. The first derivative (18) or (19) with respect to n, at $n = 0$ model give a partition sum of the closed SAW, of which the length is not fixed but controled by a fugacity x or chemical potential ($\mu = kT \log x$).

Small step fugacity describes small polymers, and critical (dilute) step fugacity the solution of a very long polymer, while a larger fugacity leads to a SAW that fills the lattice with a non-zero density.

In a poor solvent the polymer has a tendency to demix or phase separate. The phase transition between a well-dissolved polymer and a coagulated polymer is called the θ-point, and is described by the hexagonal loop model with n=1 and x=1 [20]. Of course this model itself has many loops. But at the point n=x=1 one can, for a fixed configuration of one open or closed walk, sum over all compatible configurations of the other loops. This sum results in an explicit attractive interaction between segments of the original walk which pass by the same hexagon, but are distant along the chain.

Exercise 3.8 *Show that this sum over all loops but one can be done, and determine what is the precise nature of the attractive interaction.*

In ref [20] it is argued that this model sits precisely on the θ-transition between a ordinary (swollen) SAW and a coagulated one. The argument is based on the observation that the model is critical and of a different universality class than the ordinary SAW. Then, on the assumption of the simplest possible phase diagram, the model must represent the locus of the θ-transition.

3.1.5 Hard Hexagon and Hard Square model

In the ABF model (the ADE model based on the CPL and A_m diagram on the square lattice) not all m nodes of A_m need to be distinguished. Since the lattice is bipartite, one sublattice can take only even nodes as their states, and the other only odd. This admits a simplification by identifying even and odd nodes as the same state. In A_4 for instance may we identify the nodes 1 and 4 and the nodes 2 and 3. The resulting diagram is shown in figure 3. The model can be interpreted as a lattice gas: the nodes 1 and 4 can represent an occupied site, and 2 and 3 an empty one. In this model every face can be occupied by at most one particle and two adjacent sites can not be simultaneously occupied. On the square lattice this model is called the Hard Square model[9], as the particles can be seen as tiles of size $\sqrt{2} \times \sqrt{2}$ placed in diagonal orientation with respect to the lattice. The neighbor exclusion is equivalent to the restriction that the tiles are not permitted to overlap. The critical point resulting from the ADE construction, is in the universality class of the tricritical Ising model.

Exercise 3.9 *In the Hard Square model that follows from this construction, two particles occupying second neighbor sites, have an attractive interaction; calculate the strength of this interaction.*

If one allows the weights of the two local loop configurations in the CPL model to be different, another simple model can be constructed, known as the Hard Hexagon model[10]. The meaning of the nodes on the A_4 diagram is the same: 1 and 4 signify occupied and 2 and 3 empty. By a judicious choice of anisotropic loop weights of the CPL model, one may achieve that the Boltzmann weight of the square can be factorized into the weights of two triangles with full triangular symmetry. The resulting model can be seen as a lattice gas on the triangular lattice, in which simultaneous occupation of nearest neighbors is forbidden. This is the model known as the hard hexagon model, at its critical point.

Exercise 3.10 *Verify such a model can indeed be obtained by the* ADE *construction with* A_4 *on the anisotropic CPL model. One must permit negative local loop weights, but in the resulting hard hexagon model each Boltzmann weight is positive. The triangular symmetry requires that the faces along one diagonal can not be simultaneously occupied, and that the faces along the other diagonal do not interact. Calculate the fugacity in the resulting lattice gas.*

Exercise 3.11 *Formulate arguments that the Hard Square model and the Hard Hexagon model as constructed above, must be in the universality class of the tricritical Ising model and the critical three-state Potts model respectively.*

3.2 Operators and Correlation functions in ADE models

Let the function $P(k)$ be the probability that an arbitrary face in the ADE model is in the state k. We will refer to is as the one-point distribution (1PD). Consider a loop well inside a large lattice. We assume that the 1PD is unaffected by the presence of the loop (or any other loop). In other words we assume that the 1PD conditional on the presence of a loop is the same as the unconditional 1PD. That this is plausible follows from the calculation of the partition sum above: the contribution to the partition sum of a particular domain is independent of the domains it is contained in, and it is independent of all the domains, once the state of these domains has been summed over. A stronger argument will result from the construction below.

The conditional probability $P(k|j)$ that the interior domain of a loop is in state k, provided the outside domain is in a given state j, is determined by Eq. (21) as $P(k|j) = A_{k,j}S_k/(n\,S_j)$. Thus, with the definition of the 1PD, we find the joint probability $P(k,j)$ that the outside of a loop is in state j *and* its inside in state k as

$$P(k,j) = P(j)P(k|j) = P(j)A_{k,j}\,\frac{S_k}{n\,S_j}\,. \tag{23}$$

Summation on j now yields the probability that the inside domain is in state k, which should be equal to $P(k)$:

$$\sum_j P(k,j) = P(k)\ . \tag{24}$$

The equations (23) and (24) form a consistency condition on $P(k)$. Equation (24) has the form of an eigenvalue equation for the matrix A, which immediately gives the unique solution

$$P(k) = (S_k)^2\ . \tag{25}$$

Here and below we assume that the eigenvectors are normalized, as $\sum_j (S_j)^2 = 1$.

We thus calculated the 1PD for a spin well away from the boundary. However, there is a special boundary condition which makes (25) exact for all spins. For this we a bounded lattice of arbitrary size, with the faces on the boundary all in the same state, *but with Eq. (25) as the probability distribution for that state.* Then by induction the same distribution holds for the domains separated from the boundary by a single domain wall, and so on recursively to the innermost domains. It is then assumed that in the thermodynamic limit the boundary condition should not matter, well away from the boundary.

Instead of considering the 1PD, one can take expectation values of any function of the state of one or more given faces. Let S^μ be the μ-th eigenvector of the adjacency matrix with eigenvalue Λ_μ, and S (without upper index) be the eigenvector with eigenvalue n. Then consider the function

$$w(k) = S^\mu_k/S_k\ , \tag{26}$$

If this function is an operator insertion in a correlation function $\langle \cdots S^\mu_k/S_k \cdots \rangle$, with k is the state of a given face, it effectively changes the weight of the loops *surrounding that face.* This is easily seen in the expression (21): the factor S^μ_k/S_k replaces the numerator by S^μ_k, so that the weight of the loop becomes equal to the corresponding eigenvalue Λ_μ, as long as the loop does not surround other operator insertions. We will call these functions weight-changing operators of weight Λ_μ.

A interesting result comes from the two-point function of $w(k)$.

$$\left\langle \frac{S_j^\mu}{S_j} \frac{S_k^\nu}{S_k} \right\rangle ,$$

k and j being the state of two arbitrary faces. The weights of the loops surrounding either of these faces but not the other is changed into the respective eigenvalues Λ_μ and Λ_ν, corresponding to the eigenvectors S^μ and S^ν. Now consider the innermost domain that surrounds both faces, and let j its state. After the states of all the domains nested inside it are summed over, the weight governing the state j of the final domain is

$$\frac{S_j^\mu S_j^\nu}{(S_j)^2} ,$$

aside from the weight of the domain wall surrounding it. This can be expanded as a linear combination of the other weight-changing operators:

$$S_j^\mu \, S_j^\nu = S_j \sum_\kappa C_{\mu\,\nu}^\kappa \, S_j^\kappa , \tag{27}$$

where, (again provided the eigenvectors are normalized),

$$C_{\mu\,\nu}^\kappa = \sum_j \frac{S_j^\mu \, S_j^\nu \, S_j^\kappa}{S_j}. \tag{28}$$

Apparently the combination of two operators labeled μ and ν looks from a distance like a linear combination of operators κ. This formula was first found by Pasquier[14], but is known as the Verlinde formula[15] as he found it in a more general context.

These structure constants of the operator product expansion, or fusion rules, may of course be readily calculated explicitly for any adjacency diagram, but here we only note that (i) they are symmetric in μ, ν and κ, that (ii) they take integer values for the adjacency diagram of A_m, and (iii) that they vanish if one of the indices corresponds with the largest eigenvalue, while the other two are not the same.

Exercise 3.12 *Verify (i) and (iii) of these properties, and for the A_m diagram also (ii)*

Property (iii) implies that the two-point correlation function of two different weight-changing operators vanishes in the thermodynamic limit. Obviously these structure constants may be used just as well in correlation functions of more than two operators.

3.3 The O(n) model

Another representation of the loop model on the hexagonal model, is the O(n) models with

$$Z_{\mathrm{O}(n)} = \int \mathcal{D}s \, \prod_{\langle j,k \rangle} (1 + x \; s_j \cdot s_k) \tag{29}$$

The n-component spins s sit on the vertices j of the hexagonal lattice, and the interaction factors are between nearest neighbors. The name O(n) model, refers to the global O(n) symmetry under which the integrand is invariant. The integration measure $\mathcal{D}s$ denotes integration over all spins and is normalized such that $\int \mathcal{D}s = 1$. The (mean) spin length is set by $\int \mathcal{D}s \; s \cdot s = n$. The loop model is obtained as the expansion of $Z_{\mathrm{O}(n)}$ in powers of x: The product under the integral is expanded, and the second term in the binomial is indicated as a bond on the corresponding lattice link. The only terms in that expansion that survive the integration over the spins have an even number of bonds incident in each lattice sites. The spin integration involves a sum over spin components, which results in a factor n per loop.

Exercise 3.13 *Show that the steps in the above derivation indeed work and result in the fact that $Z_{\mathrm{O}(n)} = Z_{\mathrm{hex}}$ from equations (29) and (19)*

Exercise 3.14 *Show that the square loop model can be obtained by O(n) spins on the vertices of the 4-8 lattice (a lattice with coordination number 3, where, at each vertex two octagonal and one square face meet).*

Ordinarily one is interested in the critical behavior of the standard Heisenberg model with three spin components. This has a straightforward generalization to n-component spins. The natural form of the Hamiltonian is then

$$\int \mathcal{D}s \, \exp\left(\sum_{\langle j,k\rangle} K s_j \cdot s_k\right) \tag{30}$$

on any lattice. Universality would in principle assert that this model, having the same $\mathrm{O}(n)$ symmetry, has the same critical behavior as (29), but let us look critically at the difference between the two models.

By expanding the exponent in (30) one gets multiply occupied bonds (powers of $Ks_j \cdot s_k$). One then has to sum over all ways to connect the elementary steps of the diagrams pairwise in each vertex. (see exercise below) Every closed polygon acquires a weight n by the summation over the indices. By the multiple occupancy of the bonds the polygons may cross each other as well as themselves. Thus the main differences between (29) and (30) in the high temperature (or weak coupling) limit, are (i) multiply occupied bonds, and (ii) intersections. What remains intact is the fact that the diagrams consist of closed loops, with weight n for each loop.

Let us take the premise that the model (29) is exceptional in its critical behavior. This implies that it is a fixed point in a larger parameter space, and the deviation from it is relevant. This means that such positive deviation will grow, but if it is zero it remains so.

When one looks at the graphs of both models (29) and (30) from a big distance, multiply occupied bonds can not be distinguished from neighboring parallel edges, each of which is singly occupied. This suggests that under RT multiply occupied bonds will be generated. This inconsistent with the premise that these multiply occupied bonds contribute to a relevant perturbation which would differentiate between models (30) and (29).

Thus loops seen on a great distance may begin to touch each other. However, even on a large scale such loop will not seem to intersect if they did not intersect to begin with. This suggests that intersections can not be generated by an RT, and therefore a model without intersections may well be a fixed point in the space of models where intersections are allowed. The non-intersecting loop model will be generic if the intersections are irrelevant, and exceptional if intersections are relevant. This will have to be verified.

Exercise 3.15 *Consider the O(n) spin model (29) on the square lattice, and find the set of possible diagrams in the formal expansion of Z in powers of x, and the prescription for the weight of these diagrams.*

3.3.1 Models with Cubic symmetry

The cubic symmetry in this context denotes the symmetry group of the n-dimensional hypercube. The spins can be discrete and point say to the vertices of the cube, or to the hyperfaces of the cube, but they can also be continuous, and have a weight function with cubic rather than $\mathrm{O}(n)$ symmetry, i.e. depend as a symmetric function on the components of the spin vector, rather than on only its length. As long as the partition sum is defined as in (29) (with the spin-measure adjusted), the small-x expansion is again the loop model to all orders in the expansion.

Another possibility is to place spin on the faces of the lattice: Let each of n spin components be an Ising spin, and let nearest neighbors be different in at most one of the components, at the weight cost x. Then the expansion of the partition sum in powers of x (in this case a low temperature expansion) is again a loop model, the loops signifying domain walls between domains of equal spin. Since two neighboring domains can differ in one of their n components the loop weight is again n. In fact this is a form of an ADE model, in which the graph $\mathcal{G}$is the hypercube, with vertices (corners) as nodes, and edges as links.

3.3.2 Spin correlation function

In the case of the O(n) model, the correlation functions like $\langle s_o.s_r\rangle$ will, in the small x expansion correspond to a different set of diagrams than the partition sum, rather than just a change the weights. Indeed in the expansion of $\langle s_o.s_r\rangle$ in powers of x it is clear that the number of bonds incident on o and r should be odd rather than even. An example of such diagrams is shown in figure 8. In general the diagrams will also contain closed loops, in addition to a path from o to r. Generalizations to correlation functions involving more spins or higher powers of them, are immediate. Later we will see that the diagrams are the same as the diagrams for disorder operators of ADE models.

4 The Coulomb Gas

The CG approach is based on the ADE model with state diagram A_∞. The states are called heights and traditionally take values in $\pi\mathbb{Z}$, rather than $\mathbb{Z}$. We imagine an RG transformation in which the heights are not rescaled. Let us first consider the case n=2, so that the loops can be considered domain walls surrounding a hill or a valley both with equal weight.

On a large scale the average heights will take continuous values with possibly a preference for integer multiples of π. Since the renormalized Hamiltonian can only depend on differences or gradients of h, it should be expandable in powers of the gradient:

$$H = \int \mathrm{d}^2 r\ \frac{g}{4\pi}(\partial h)^2 + g_2(\partial h)^4 + \cdots - z_2 \cos 2h - z_4 \cos 4h - \cdots \tag{31}$$

We consider even powers only, because the Hamiltonian has reflection symmetry in space. Because, in general the Hamiltonian contains terms periodic in h, we define a Renormalization such that h is invariant. This implies that the free field Hamiltonian, with only the first term, is unchanged under RG, which follows from simple power counting. By the same argument the higher order powers in ∂h will be irrelevant (but may renormalize the leading term). To inspect the relevance of the parameters z_j we can calculate the correlation function

$$\langle\ \cos(eh_o)\ \cos(eh_r)\ \rangle = \frac{1}{2}\langle\ \exp(\mathrm{i}eh_o - \mathrm{i}eh_r)\ \rangle \tag{32}$$

with some prefactor e (not to be confused with the base of the natural logarithm e). To calculate the r dependence, we need to do the integral

$$\int \mathcal{D}h\ \exp\left(\int \mathrm{d}^2 r\left[-\frac{g}{4\pi}\ (\partial h)^2 + \mathrm{i}eh_o - \mathrm{i}eh_r\right]\right) \tag{33}$$

a Gaussian integral. Since $\partial\cdot\partial \log r = 2\pi\delta(r)$, the result is

$$\langle\ \cos(eh_o)\ \cos(eh_r)\ \rangle\ \propto\ \exp\left(\frac{-e^2}{g}\log|r|\right) = |r|^{-e^2/g} \tag{34}$$

The middle expression explains the notation e for the prefactor: the operators behave as particles with electric charge e.

Exercise 4.1 *Verify (34) by performing the Gaussian integration in (33)*

The $\cos(2h)$ term in (31) is relevant if the correlation function decays slower than $|r|^{-4}$, i.e. when $g > 1$. The higher harmonics will be less relevant, so we need only consider them if for some special reason z_2 vanishes.

Thus starting from the Hamiltonian (31) of a fluctuating height field, we see emerge a model of interacting electric charges, hence the name "Coulomb Gas". It is the preference for discrete values of the height that causes the presence of the charges. We already noted the relevance of the charge fugacities at sufficiently large g. This can be understood intuitively from the fact that large g implies that the h-field tends to be flat, and thus tends to be pinned to the discrete values. This phase is not critical. When g is small, and z_2 is irrelevant, the model renormalizes to a purely Gaussian model, with the result that the correlation function (32) decays algebraically (34), indicating that the model is critical.

We noted before that the parameter g is marginal on the basis of power counting. But this argument is only valid at the line of fixed points, where all z_e vanish. Otherwise the parameter g will renormalize, as is calculated approximately in [5]. But the only thing we need is the sign of this renormalization. This can be reasoned easily: when charges are present they will screen the interaction between them. Therefore g will increase when z_2 is positive. When positive and negative charges are present in different abundances, say with fugacities $z{+}2$ and $z{-}2$, the contribution to the partition sum and to the renormalization of g comes in lowest order from the neutral product $z{+}2\, z{-}2$.

Because g renormalized as a result of the presence of charges and of terms in the Hamiltonian of higher order in ∂h, we do not know *a priori* the renormalized value of g, that results from the original ADE model or the loop model. It is evident that g will be a decreasing function of the original step fugacity. When the step fugacity is small almost all heights will be the same: the smooth phase. If the domain walls are sufficiently numerous, the height fluctuations become more probable and it is reasonable that eventually g will sink under the threshold value 1. We can, however, conclude that everywhere in the rough phase, the exponent x_e of the correlation function

$$\langle\, \cos(eh_o)\ \cos(eh_r)\,\rangle \ \propto\ |r|^{-2x_e} \tag{35}$$

is proportional to e^2, and that at the roughening transition the prefactor is $1/2$.

4.1 Vortices

In some of our applications we will need the behavior of vortices or screw dislocations in the height model. These are points in the height configuration around which $\oint \partial h \cdot \mathrm{d}r$ is non-zero. While in the A_∞ model they only exist as insertions, these operators have a natural abundance in systems in which the height is defined on a periodic interval. In order to calculate the exponent of vortices, it is more convenient to consider two vortex insertions, in the absence of other vortices or periodic fields. A vortex can be seen as a source or sink of domain walls.

Consider thus the integral

$$\int \mathcal{D}h\ \exp\left(-\int \frac{g}{4\pi}\,\mathrm{d}^2r\ (\partial h - 2\pi k)^2\right) \tag{36}$$

Here we take k a fixed vector field, zero almost everywhere, and the value m on a line that connects two vortex insertions, in the direction orthogonal to the line. We use the continuum space notation because it is more compact, but it is convenient to think of the ∂'s as lattice differences, and the spatial integral as a sum over the lattice. However, we may 'sum by parts' just as we may integrate by parts. $\int \mathrm{d}^2r A\partial B = -\int \mathrm{d}^2r B\partial A$ provided AB vanishes on the boundary of the integration

We have a Gaussian integral with terms in the exponent, quadratic or linear in h. This can be done quickly after the introduction of the notation G for the inverse of the Laplace operator: $\partial \cdot \partial G(r-r') = 2\pi\delta(r-r')$. Asymptotically G differs from the continuum version $\log(r-r')$ only by a constant term and terms of order $\mathcal{O}(r^{-2})$.

The action then is transformed successively as (with the Einstein summation convention)

$$\Big(-\frac{g}{4\pi}\,(\partial_i h - 2\pi p_i)^2\Big) \;\;\rightarrow$$

doing the spatial integration (or rather summation) by parts

$$\Big(\frac{g}{4\pi} h\partial_i\partial_i h - gh\partial_i p_i - \pi\; g\; p_i p_i\Big) \;\;\rightarrow$$

Gaussian integration over the height field, and insertion of the spatial identity kernel in the last term

$$\frac{g}{2}\Big(\,-(\partial_i p_i)G(\partial_j p_j) - p_j(\partial_i\partial_i G)\; p_j\Big) \;\;\rightarrow$$

partial integration and commutation of G with ∂

$$\frac{g}{2}\Big(p_i(\partial_i\partial_j G)p_j + (\partial_i p_j)G\;(\partial_i p_j)\Big) \;\;\rightarrow$$

same thing, now only in the first term

$$\frac{g}{2}\Big(\,-(\partial_j p_i)G(\partial_i p_j) + (\partial_i p_j)G(\partial_i p_j)\Big) \;\;\rightarrow$$

(backwards) rewriting of the product of the antisymmetric tensors

$$\Big(\frac{g}{2}(\varepsilon_{ij}\partial_i p_j)G(\varepsilon_{kl}\partial_k p_l)\Big)$$

After all this, noting that $(\partial \times k) = (\varepsilon_{ij}\partial_i p_j)$ vanishes everywhere, except at the vortices, where it takes the value m and $-m$ repectively, we conclude that

$$\int \mathcal{D}h\;\exp\left(\int \mathrm{d}^2 r\;-\frac{g}{4\pi}\,(\partial h - 2\pi k)^2\right) \propto |r|^{-gm^2} \tag{37}$$

This result is very much like the correlation of two cosine functions of h, but here the exponent is proportional to g. They are viewed as magnetic rather than electric charges.

The interaction between electric and magnetic charges is fairly obvious: when an electric charge is taken around a vortex, it must gain a phase factor equal to $\exp(2\pi \mathrm{i} e m)$, so that the interaction will look like

$$\left(\frac{z}{|z|}\right)^m \tag{38}$$

in terms of the complex number $z = r_\mathrm{x} + \mathrm{i} r_\mathrm{y}$ where r is the spatial vector between the two vortices.

4.2 Choosing the loop weight

When we want to consider loop models with general n, we can still make use of the A_∞ model, but select a subdominant eigenvalue, say, $n = 2\cos\pi e_o \leq 2$. The corresponding eigenvector is $\exp(e_o \mathrm{i} h)$. With this information we can follow the protocol of Sec.3.1 and construct the weights of the height model. Each bend of the loop with angle γ_b with the domain with the higher h on the inside, has a weight $\exp\left[\frac{e_o \mathrm{i}\gamma_\mathrm{b}}{2\pi}\right]$. When the lesser height is on the inside the weight is conjugated.

As an aside, it is an interesting game,

Exercise 4.2 *or a difficult exercise*

to note that this addition in the continuum contributes to the action the term

$$\frac{\mathrm{i}e_0}{2\pi}\left(\frac{(\partial_i h)\ (\partial_i\partial_j h)\ (\partial_j h)}{(\partial_k h)\ (\partial_k h)} - \partial_i\partial_i h\right)\ . \tag{39}$$

This expression measures the curvature density of the contour lines (level sets) of h, with a sign so that a hill is circumscribed by positive level sets and a valey by negative. This of course is precisely what one expects the term to do in the continuum. The second term in (39) is a full derivative, but it will contribute unless the boundary condition makes ∂h vanish at the boundary. Integrated over the entire configuration (39) takes the form of the sum of the values of h at the extrema, minus the sum of h at the saddle points minus twice h at infinity. This is fairly obvious when it is taken into account where the expression comes from, because this counting of extrema and saddle points counts precisely the domain walls with a sign for their orientation.

Now consider the model on a bounded domain, and take the heights at the boundary all equal to 0. This boundary condition precisely reproduces the loop model, with the condition that the loops are not permitted to penetrate the boundary. Now we introduce the weight-changing operator (26), based on the eigenvector $\exp(-e_o\mathrm{i}h)$, with eigenvalue n, and name it $\mathcal{B}$. When we insert $\mathcal{B}$ at some position in the lattice, it has no effect on the weights of the loops in the configuration, because the changed weight is the same as the original. This observation is independent of the size and shape of the domain or the locus of the insertion. The operator $\mathcal{B}$, however is not the identity operator. In particular, it must not be inserted twice, in the same or in different places, since that changes the weight of the loops that surround both insertions. When expressed directly in the height variables the operator $\mathcal{B}$ has the form

$$\mathcal{B} = \exp\left(-2\mathrm{i}e_o h\right)\ , \tag{40}$$

an electric charge of strength $-2e_o$. It is sometimes called the background charge, or charge at infinity. The loops surrounding a hill have weight $\exp\left(e_o\mathrm{i}\pi\right)$, while the loops that surround a valley have weight $\exp\left(-e_o\mathrm{i}\pi\right)$. For the loops that surround the operator insertion these two weights are interchanged. This can be taken to mean that for those loops inside and outside are interchanged. In other words, the operator insertion plays the role of the point at infinity.

In summary we note that there are two alternative ways to calculate the partition sum: with and without the insertion of the background charge. The same applies to correlation functions. This will play an important role in the sequel.

4.3 Weight-changing operators

If we want to construct a weight-changing operator with weight Λ_e, we select the eigenvector of the adjacency diagram A_∞ with that eigenvalue, say $\exp(\mathrm{i}eh)$ or equivalently $\exp(-\mathrm{i}eh)$, with $\Lambda_e = 2cos(\pi e)$. In the corresponding two-point function

$$\langle \exp(\mathrm{i}\,e\,h_o - \mathrm{i}\,e_o\,h_o)\ \exp(-\mathrm{i}\,e\,h_r - \mathrm{i}\,e_o\,h_r)\rangle \tag{41}$$

we combine the complex conjugate pair, and as in (26), divided by the eigenvector with eigenvalue n. This immediately determines the exponent of the operator

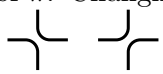
$$x_e = \frac{e^2 - e_o^2}{2g}\ . \tag{42}$$

4.4 Application to the Potts model

In this section we will show that the CG predicts the critical exponents of the Potts model. First we remember that the loop weight for the q-state Potts model is $\sqrt{q}$, and that the background charge follows from $\sqrt{q} = 2\cos(e_o\pi)$ (up to a sign and additive multiples of 2). We will first concentrate on the thermal or energy exponent.

4.4.1 The energy operator

We have seen that the Potts model has weight $1+\sqrt{q}$ for two neighboring equal spins and weight 1 for unequal spins. Changing the temperature of the Potts model would result in a change of the ratio of these weights. The operator conjugate to the (inverse) temperature is the energy operator $\mathcal{E}$.

To see what $\mathcal{E}$ looks like in the loop language. We should note that the Potts spins live on only one of the two sublattices of the faces of the CPL model. We place the Potts variables on the sublattice where the heights are an even multiple of π and the neutral state corresponds to the odd multiples of π. Changing the temperature results in a change in the relative weight of the two diagrams  . Thus the energy operator gives an opposite sign to weight of these two diagrams. What does this do the the weight of the total configuration? Note that the Potts sites are always surrounded by an even number of loops. If we concentrate on the center of the two diagrams, this too must be surrounded by an even number of loops when the two Potts variables are 'connected', and an odd number otherwise. In the latter case it has a negative weight. As a result, we can identify $\mathcal{E}$ with a weight-changing operator of weight $-n = -\sqrt{q}$.

In retrospect, it is not unexpected that $\mathcal{E}$ is this weight-changing operator as only the eigenvalues $\pm\sqrt{q}$ have an eigenvector which is symmetric under permutation of the q states. This makes this operator the natural candidate.

We will now use this to see what $\mathcal{E}$ looks like when expressed in the heights. Eigenvectors with eigenvalue $-\sqrt{q}$ are $\exp(\mathrm{i}h - \mathrm{i}e_o h)$, $\exp(-\mathrm{i}h + \mathrm{i}e_o h)$, $\exp(\mathrm{i}h + \mathrm{i}e_o h)$, and $\exp(-\mathrm{i}h - \mathrm{i}e_o h)$. In the two-point function a complex conjugate pair should be chosen so that the total charge vanishes, and the loops surrounding both operator insertions have the appropriate weight $\sqrt{q}$. The ones with the smallest charge would be the natural choice, so in the convention that e_o is positive, this would be the first two. However, in table 1, we list a number of possible height configurations around a single vertex. While each individual face has integer heights (up to

config.	$\begin{matrix}0 & 1\\ 1 & 0\end{matrix}$	$\begin{matrix}0 & 1\\ 1 & 2\end{matrix}$	$\begin{matrix}2 & 1\\ 1 & 2\end{matrix}$	$\begin{matrix}2 & 3\\ 1 & 2\end{matrix}$	$\begin{matrix}2 & 3\\ 3 & 2\end{matrix}$
height	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$
Boltzmann	$2\cos(e_o\pi/2)$	1	$2\cos(e_o\pi/2)$	1	$2\cos(e_o\pi/2)$
$\mathcal{E}$	$2\mathrm{i}\sin(e_o\pi/2)$	-1	$-2\mathrm{i}\sin(e_o\pi/2)$	1	$2\mathrm{i}\sin(e_o\pi/2)$

Table 1: The weight of local configurations. The first line shows the heights around a single vertex up to a factor π. The second line the average height of these four faces. The third line is the Boltzmann weight in the partition, and the fourth line the weight when an energy operator is inserted at the position.

factor π), the mean height at a vertex can be half-integer. This permits us to differentiate between the various eigenvectors. The third line in table 1 shows the Boltzmann weight of the corresponding configuration, and the fourth line the factor by which it should be multiplied when an energy operator is inserted. This factor should be one of the eigenvectors given above, divided by the original $\exp(\mathrm{i}e_o h)$ which generates the loop weight. Then we see that the sign of the imaginary terms in the third line of the table, does not agree with the first candidate eigenvector, but does with the third. In a two-point function it should be combined with the fourth, so that the weight of the loops surrounding both $\mathcal{E}$ insertions is $\sqrt{q}$ again. In terms of total charge, the operator $\mathcal{E}$ thus corresponds to a charge 1 or $-1-2e_o$, i.e.

$$\mathcal{E} = \exp(\mathrm{i}h) + \exp(-\mathrm{i}h - 2\mathrm{i}e_o h) \; . \tag{43}$$

Having specified the nature of $\mathcal{E}$, we are now in the position to calculate its two-point function $\langle \mathcal{E}_o \mathcal{E}_r \rangle$. Substituting (43) there are at first sight three candidates for the two-point function, i.e. the following charge pairs: (1,1), ($-1-2e_o$, $-1-2e_o$), and (1, $-1-2e_o$). To verify if these charge pairs give the appropriate weight to the diagrams, note that the weight of each loop is given by $2\cos(e\pi + e_o\pi)$, where e is the total charge enclosed. The first candidate correctly gives the weight $2\cos(\pi + e_o\pi) = -\sqrt{q}$ for the loops that surround one of the two $\mathcal{E}$ insertions, and $2\cos(2\pi + e_o\pi) = \sqrt{q}$ to those loops that surround both. The second candidate gives weight $2\cos(-e\pi - e_o\pi) = -\sqrt{q}$ to the loops that surround a single insertion, but the loops that surround both will have weight $2\cos(3\pi e_o)$, which is incorrect. The last option works fine: both $2\cos(\pi + e_o\pi)$ and $2\cos(-\pi + e_o\pi)$ are equal to $-\sqrt{q}$, while the loops surrounding both insertions get $2\cos(-e_o\pi) = \sqrt{q}$.

Note that in the case of the last charge pair the total charge is $-2e_o$, which is that of the $\mathcal{B}$ operator, and should thus be considered as neutral. As a result the corresponding two-point function is simply the Boltzmann weight of the two attracting charges:

$$\langle \, \mathcal{E}(o) \, \mathcal{E}(r) \rangle \; \propto \; |r|^{\frac{-1-2e_o}{g}} \; , \tag{44}$$

analogous to (34). This must give the correct decay of the energy two-point function. The coefficient, however, can not be obtained with these simple arguments.

The third candidate for the two-point function works equally perfect for the loop weights. However, in the CG language it has the difficulty that it is not charge neutral. We have learned to accept configurations of a total charge $-2e_o$, but not 2. This problem, however can be corrected by noting that charges of magnitude -2 are freely available in the system. The interaction between the two positive unit charges can be screened by a free floating double negative charge. This leads to:

$$\langle \, \mathcal{E}(o) \, \mathcal{E}(r) \rangle \; \propto \; \int \mathrm{d}^2 r' |r|^{\frac{1}{g}} \; |r - r'|^{\frac{-2}{g}} \; |r'|^{\frac{-2}{g}} \; \propto \; |r|^{2-\frac{3}{g}} \; , \tag{45}$$

the result being obtained from simple power counting. Depending on the value of g, as yet unspecified, one may have to regularize the UV singularities of the integral. If in the scaling limit $\mathcal{E}$ consists of positive charges only, they must be prevented to form a unit negative charge by picking up a -2.

Since these two calculations of the two-point function, (44) and (45) represent exactly the same loop weights in the corresponding loop correlation function, they must be equal. This consistency condition gives a relation between g and e_o:

$$\frac{-1-2e_o}{g} = 2 - \frac{3}{g} \quad \text{so that} \quad e_o = 1 - g \; . \tag{46}$$

This argument resolves the uncertainty of the value of g.

An independent argument[21, 22] to relate g and e_o is based on the role of the screening charges introduced in (45). If these operators are relevant their ubiquitous presence would drive the system away from the dilute CG. However, if they are irrelevant they would be ineffective at a sufficiently large spatial scale. Therefore, for the CG description to be applicable, they must be marginal. Remember that the two-point function of a marginal operator decays as r^{-4}. The screening charge -2 can be viewed as a neutral weight changing operator, associated with the eigenvector $\exp(-2\mathrm{i}h+\mathrm{i}e_o h)$. The exponent follows from (42) $x = (-2)(2-2e_o)/g = 4(e_o-1)/g$. The requirement that this be -4 leads to the same relation (46).

Exercise 4.3 *A third way to calculate the two point function is to take the integral (45), and insert $\mathcal{B}$ at an arbitrary fixed position. Try to show that this integral again leads to the same result, now assuming (46) from the outset. Hint: note that the integrand behaves conveniently simple under Moebius transformations, i.e. $z \to (a+bz)/(c+dz)$ with $ad-bc=1$, in complex coordinates. Then judiciously choose a Moebius transformation that simplifies the integral.*

In conclusion we have determined the thermal exponent of the q-state Potts model as

$$x_\varepsilon = 1 - \frac{3}{2g}, \quad \text{where} \quad \sqrt{q} = -2\cos\pi g, \quad \text{and} \quad g \leq 1 \tag{47}$$

The last condition goes back to the requirement that the charges ± 2 should be irrelevant, and the for $\sqrt{q}$ we need the positive root.

Exercise 4.4 *Calculate the exponents α and the ν of the three-state Potts model*

4.4.2 Critical and tricritical and antiferromagnetic points

The Potts model is defined for any positive integer q. Its low or high temperature series expansion even for all q. It has been known for a long time that the transition is continuous for $q \leq 4$ and first-order for $q > 4$.[23] The transition point is an analytic curve for all $q \neq 0$, but the analytic continuation of the thermal exponent beyond $q = 4$, folds back to smaller q, see (47). This suggests that it may describe critical behavior of another kind in a model with the same S_q symmetry. Continuity of the RG picture indicates that the other kind of critical behavior is a tricritical point[24], i.e. a special critical point where the transition changes from continuous to first-order.

When we constructed the Potts model with the ADE construction, we noted that in the case of the dense or dilute loop models, the central node of the diagram plays the role of a vacancy or unoccupied site. In principle the introduction of a vacancy as allows one to drive the transition discontinuous: if the vacancy has large Boltzmann weight, and the Potts variables on the other hand have a strong interaction, there must be a first-order transition between the S_q symmetric dilute phase, and a $\mathrm{S}_{q\text{-}1}$ symmetric ordered phase. Thus, in a dilute Potts model one may expect a critical and a tricritical RG fixed point, as shown in figure 4. The observations above suggest that the critical and tricritical fixed points are the analytic continuation of each other, meeting at $q = 4$. This gives us an interpretation for another branch of Eq.(47) for $g \geq 1$. At this branch, the condition that the double charges are irrelevant is no longer necessary, because in order to reach the tricritical point one has to tune another variable, which precisely suppresses the double charges. However, from the asymmetry between positive and negative charges, noted before, the +2 and -2 charges can not be both suppressed by tuning a single parameter. Only the positive ones are suppressed at the tricritical point, leaving the negative ones as screening charges. This leads to a crossover exponent, which generates the RG flow from the tricritical to the critical point (figure 4). The operator associated with this flow must be the presence of positive double charges. Like in the case of the leading thermal exponent, this crossover exponent can be calculated from two constructions of the two-point function. One is by letting

two double positive charges be screened by two negative ones, and the other is by the simple correlation of a $+2$ with $-2-2e_o$. As in the case of the leading thermal exponent, the result is the same.

$$x_{\varepsilon,2} = 4 - 2g \quad \text{with} \quad g \leq \frac{3}{2} \tag{48}$$

Exercise 4.5 *Show that both constructions result in this same exponent.*

This is the exponent that governs the RG flow on the transition manifold from the tricitical into to the critical regime, and further on into the critical fixed point (where it has a smaller value g and is irrelevant).

The critical Boltzmann weight of the Potts model can be analytically continued beyond $q = 0$, as seen in 3.1.3. It is in the antiferromagnetic domain, where unequal spins have higher weight than equal spins. Having calculated the leading thermal exponent we see that in this AF regime, the temperature is an irrelevant parameter, and there should be a whole critical interval (in T), attracted to what is the analytic continuation of the FM critical point. This critical phase is known as the Berker-Kadanoff[31] phase.

4.4.3 The energy four-point function

The critical two-point functions are simple power laws, and thus contain no information beyond the value of the exponent. Also three-point functions, given that they are conformally invariant, are completely determined by their exponents[32]. Four-point functions, however, are non-trivial. Here we sketch the way how to calculate them, mainly to demonstrate the power of the CG approach. We use complex numbers to denote positions in the plane.

Already when we considered the two-point function, we noted several ways to calculate it. Here too, we can take different approaches. One is to take four positive unit charges to represent the operators $\mathcal{E}$. They should be screened by two double negative charges. The resulting fourfold integral

$$\Gamma(r_1, r_2, r_3, r_4) \equiv \langle \mathcal{E}(r_1)\mathcal{E}(r_2)\mathcal{E}(r_3)\mathcal{E}(r_4)\rangle =$$

$$\prod_{1\leq j<k\leq 4} |r_j - r_k|^{1/g} \int \mathrm{d}^2 w_1 \, \mathrm{d}^2 w_2 \; |w_1 - w_2|^{4/g} \prod_{j=1}^{2}\prod_{k=1}^{4} |w_j - r_k|^{-2/g} \tag{49}$$

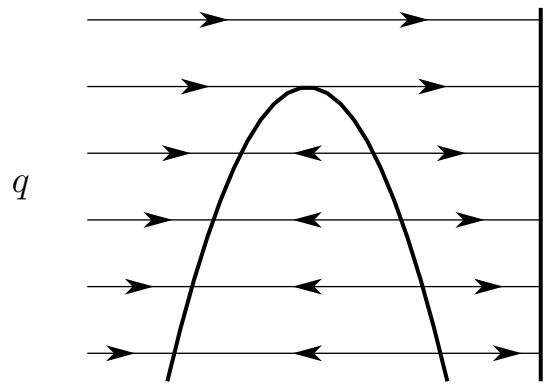


Figure 4: The RG flow on the transition sheet of the dilute Potts model. The parabolic bold line is the line of critical and tricritical fixed points, and the vertical bold line represents the line of first order fixed points.

is not very attractive to even attempt. It becomes simpler if one of the $\mathcal{E}$ is combined with a $\mathcal{B}$, so that quasi-neutrality is achieved with a single screening charge. The result is

$$\Gamma(r_1,r_2,r_3,r_4) = \prod_{1\le j<k\le 3} |r_j - r_k|^{1/g} \prod_{k=1}^{3} |r_k - r_4|^{2-3/g} \;\times$$

$$\int \mathrm{d}^2 w \;\; |w - r_4|^{6/g-4} \prod_{k=1}^{3} |w - r_k|^{-2/g} \tag{50}$$

One may verify that the integrand is invariant for Moebius (see exercise above eq.(47)) transformations, which can be used to simplify the expression. The area integral can reduced to a contour integral, making use of the following theorem using complex notation for the plane. Let $p(u,v) = \partial/\partial v P(u,v)$ be functions analytic in a domain D and $\bar{D}$ for their two arguments respectively, then

$$\int_D \mathrm{d}^2 z \; p(z,\bar{z}) = \frac{1}{2\mathrm{i}} \oint_{\partial D} \mathrm{d}z \; P(z,\bar{z}). \tag{51}$$

To apply this to (50) we take D to be complex plane with appropriate cuts, to exclude non-analyticities due to the fractional powers. The calculation[25] requires some bookkeeping, and the result is, expressed in the complex variables $u_i = (r_i - r_4)(r_j - r_k)$ in which the indices i,j,k are a cyclic permutation of $1,2,3$.

$$\Gamma(r_1,r_2,r_3,r_4) = -\frac{1}{2}\left[\left(1 + 2\cos\frac{2\pi}{g}\right)|G|^2 + |H|^2\right]\tan\frac{\pi}{g} \tag{52}$$

in which the contributions G and H are expressed in the hypergeometric function $\mathrm{F}_{2,1}$ and the Beta function B.

$$\begin{aligned} G &= \mathrm{B}\big(\tfrac{3}{g}-1,\; 1-\tfrac{1}{g}\big)\left(\frac{u_1}{u_2}\right)^{1/2g} u_3^{1-3/2g}\,\mathrm{F}_{2,1}\left(\tfrac{1}{g},\; 1-\tfrac{1}{g};\; \tfrac{2}{g};\; -\frac{u_1}{u_2}\right) \\ &\text{and} \\ H &= \mathrm{B}\big(1-\tfrac{1}{g},\; 1-\tfrac{1}{g}\big)\left(\frac{u_1 u_3}{u_2}\right)^{1-3/2g}\mathrm{F}_{2,1}\left(2-\tfrac{3}{g},\; 2-\tfrac{1}{g};\; 2-\tfrac{2}{g};\; -\frac{u_1}{u_2}\right) \end{aligned} \tag{53}$$

Exercise 4.6 *Verify that this four-point function is covariant under conformal transformations, as follows:*

$$\Gamma[f(r_1),f(r_2),f(r_3),f(r_4)] = \Gamma(r_1,r_2,r_3,r_4)\prod_{i=1}^{4}|f'(r_i)|^{-x_\varepsilon}\;, \tag{54}$$

where $f(r)$ is a conformal (analytic) transformation, and f' its derivative.

We stress that conformal invariance is a result, and not input in the calculation.

An interesting result of this calculation, even irrespective of the explicit outcome, is the prediction that the integrals (49) and (50) are the same up to an overall factor (which may actually be calculable within the scheme).

4.4.4 The magnetic exponent, spin and disorder operators

The adjacency matrix of the Potts model has eigenvalues $\sqrt{q}$, $-\sqrt{q}$ and 0, the latter (q-1)-fold degenerate. The first two are symmetric under the group S_q of permutations, but the last $q-1$ break that symmetry. The weight-changing operator associated with the second eigenvalue is the energy, as noted above. An example of an eigenvector with eigenvalue 0 is $\exp(2\pi \mathrm{i} s/q)$

for $s \in \{1, 2, \cdots, q\}$ and 0 in the symmetric state $s = 0$. The corresponding weight-changing operator represents what is called the spin operator of the Potts model $\mathcal{S}$. In its two-point function it gives weight zero to all loops that surround one of the insertions and not the other. Only those domain-wall configurations contribute in which the two insertions are in the same domain.

Exercise 4.7 *Verify that this two point function is, up to an additive and a multiplicative constant equal to the probability that two spins are in the same state*

Can we find a specific charge operator in the CG representation, that does precisely that? We have already seen that when we fix all the heights at the boundary of the system to zero, each loop has a weight $2\cos(e\pi + e_o\pi)$, where e is the total charge enclosed. Since this weight must be zero for the operator $\mathcal{S}$, we choose $e_\pm = -e_o \pm 1/2$. Since $e_+ + e_- = -2e_o$, the weight of the loops surrounding both insertions remains $\sqrt{q}$.

The exponent of the corresponding two-point function is[26]

$$x_s = \frac{e_+\, e_-}{2g} = \frac{\frac{1}{4} - (1-g)^2}{2g} \tag{55}$$

Exercise 4.8 *Verify this result and try to argue that there are other contributions to the correlation function with exponent* $[9/4 - (1-g)^2]/(2g)$.

In the Potts model we may also introduce what is called the disorder operator $\mathcal{D}$, which is the terminal of a cut across which the Potts spins s are identified with $P\,s$, for a non trivial permutation $P \in \mathrm{S}_q$. The disorder operator is dual[1] to the spin operator $\mathcal{S}$. That this is so, can be seen e.g. as follows. Above we argued that the only contributions to its two-point $\mathcal{S}$-funtion, comes from those domain wall configurations in which the two $\mathcal{S}$-insertions are in the same domain.

Now consider the insertion of two disorder operators in two different faces. For consistency their state must be the central node of the adjacency diagram, i.e. the neutral state. There is then a cut between the two faces across which s is identified with $P\,s$. Let is for the moment assume there is no state for which $P\,s = s$ aside from the neutral state. Then any spin domain that surrounds either of the two $\mathcal{D}$-insertions and not the other has zero weight. If there is no such cluster, the two faces are in the same neutral cluster, and then the weight is not affected by the operator insertions. Clearly the correlation function is the same as that of two spin operators, with the interchange between the neutral and the spin clusters. The restriction that there is no state for which $P\,s = s$, can be lifted when explicitly the unconnected part of the correlating is deducted.

4.4.5 Anyons

An anyon, or parafermion[33] is an operator with the property that two insertions of it pick up a non-trivial phase factor if one of them is wound around the other. For a boson the phase factor would be 1, and for the fermion -1. For the any-on this phase factor could be anything. The combination of a spin operator and a disorder operator on adjacent faces turns out to form an anyon operator.

Consider a disorder operator $\mathcal{D}$ associated with the symmetry operator $P \in \mathrm{S}_q$, and a spin operator $\mathcal{S}$ chosen to be a unitary representation of P, so that $P\mathcal{S} = \mathrm{e}^{2\pi\mathrm{i}\sigma}\mathcal{S}$, with σ a rational number. We consider the operator that results from the insertion of $\mathcal{S}$ and $\mathcal{D}$ on adjacent faces. We will show that this combination behaves as an anyon.

[1] A dual transformation interchanges the strong and weak coupling regime. In the case of the Potts model, it interchanges the two sublattices which house the central state, and the q equivalent states, respectively.

To inspect the properties of this operator, we consider the two-point function, of two complementary anyons. Their disorder component is associated with the permutation P and P^{-1} respectively, and their spin insertions are each others complex conjugate. The spin operators are placed in a domain with spin states, and the disorder operators in a domain in the neutral state. The disorder operators are connected by a cut, indicated by the dashed line, across which the the state s on one side is identified with $P\,s$ Kon the other. Figure 5 shows an example. The position of the anyons is indicated with the small diamonds.

Since, as argued above, the the two spin operator must sit in the same domain, and likewise the two disorder operators, their combined positions must be on the same domain wall. The domain wall that connects the two anyons is indicated by the bold loop. The spins in the domain which it encloses should all be in the same state as far as it lives on one side of the cut. However, where it crosses the cut, this spin state s is replaced by $P\,s$. For each time the domain crosses the cut in the same direction, on its way between the two $\mathcal{S}$-insertions, the spin operators generate a phase factor $\exp(2\pi\mathrm{i}\sigma)$. Each passage in the opposite direction gives the inverse phase factor. This gives the selection rule and the weight of the domain configurations contributing to the two-anyon correlator.

Can we accommodate these rules in the height representation of the model?[27] The restriction that the two operators must sit on the same domain wall, can be represented by screw dislocations or vortices of strength 1 and -1. The domain wall that joins the two anyons reverses its signature as it passes on of the anyons. This causes the curvature of the two halfs of the domain wall to count in opposite ways. The total bending angle no longer adds up to 2π, but counts the number of times the domain walls wind around the anyons. In the figure the domain walls wind twice around the right hand anyon before reaching it. The weight of the domain wall is therefore not equal to $\exp(\pm\mathrm{i}\pi e_o)$, has a factor $\exp(2\mathrm{i}\pi e_o)$ for each time it winds the insertions. This phase factor can be cancelled by placing a factor $\exp(-\mathrm{i}e_o h)$ in each of the anyons, since the effective height at the anyon position is increased or decreased by such winding.

Exercise 4.9 *At a screw dislocation the height is by definition ill-defined. But one can define an effective height by integrating steps in height along some fixed line out to the boundary or to infinity. Show that spiraling of the domain walls that terminate in a dislocation increases or decreases this height.*

So far the screw dislocations have selected the proper diagrams, and the charges $-e_o$ have neutralized the phase factor that would otherwise be associated with the net curvature of the domain walls. Above we have seen that the diagrams do pick up a phase factor $\exp(\pm 2\pi\mathrm{i}\sigma)$

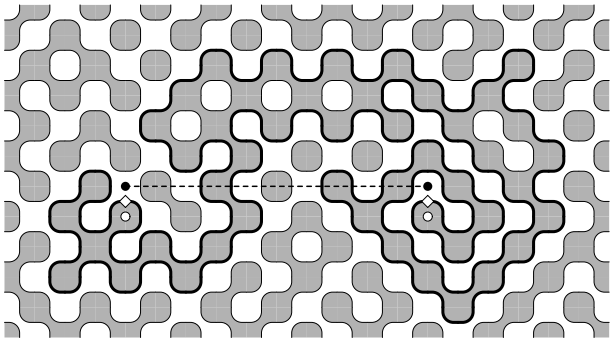


Figure 5: A configuration contributing to a two-anyon correlation function. The gray domains contain the Potts spins, and the white domains the neutral state. The black dots are the disorder operators, and the open circles the spin operators. The dashed line is the cut associated with the disorder operators.

each time the cluster passes through the cut from above or from below respectively. This phase factor can be accommodated by placing a charge $e = \pm\sigma$ in each of the anyons.

As a result, the exponent that governs the spatial decay of the two-point function is given by

$$x_{\mathrm{a}} = \frac{g}{2} m_1 m_2 + \frac{1}{2g} e_1 e_2 = \frac{g}{2} + \frac{\sigma^2 - e_o^2}{2g} = 1 + \frac{\sigma^2 - 1}{2g} \tag{56}$$

In addition these anyons have spin, i.e. they pick up a phase factor when they are taken around each other. This is caused by the interaction between the magnetic and electric charges

$$s_{\mathrm{a}} = \frac{1}{2}(m_1 e_2 + e_1 m_2) = \sigma \tag{57}$$

In principle we have now expressed the exponents of these anyons directly in their phase factor. However, σ can not be chosen to be anything. In particular since $P^k = 1$ implies that $k\sigma$ is an integer, the denominator of σ is the order of some permutation $P \in S_q$.

4.4.6 Perspective

Before we continue, we wish to take stock of the consequences of these calculations and of their qualitative assumptions.

1. The RG is taken as valid.

2. It is assumed that the critical point of the Potts model approaches the Gaussian free field model under progressive renormalization. While one can justify that the infinity of operators that separate the Potts models from the Gaussian free field (with screening charges and background charge) are irrelevant, *at the Gaussian model itself*, this says nothing about the possible occurrence of other fixed points which prevent the critical Potts model to renormalize to the Gaussian model.

3. The value of e_o follows directly from the value of q. But the coupling constant g could be determined only by indirect consistency arguments.

4. We proposed that the energy operator of the Potts model is a particular selection of weight changing operators, which corresponds to the insertion of only positive unit charges. This implies that the temperature parameter of the Potts model traces out the special line where only positive unit charges are present. This conclusion will play a role in the next section. In principle it would be possible to allow a small admixture of the other candidate weight-changing operators. This would spoil the argument that determines g, leaving it without consistent solution.

5. As in the discussion of RG, the proof of the pudding is in the eating. The many results of this approach prove it to be correct, in those cases where other reliable results are known.

4.5 The O(n) spin model

In the case of the Potts model the critical point was known beforehand, (by duality) which was a significant advantage. In the O(n) spin model (29) no such prior result existed when the CG approach was brought to bear on the problem. In this section we will deal with precisely that problem. By using (29) as a starting point, we have already resolved a few difficulties. Every link can be occupied by at most one bond, and because the lattice is three-coordinated, intersections are not possible. The intention of this section is to make use of the knowledge we have acquired about the Potts model, to apply it to the O(n) model.

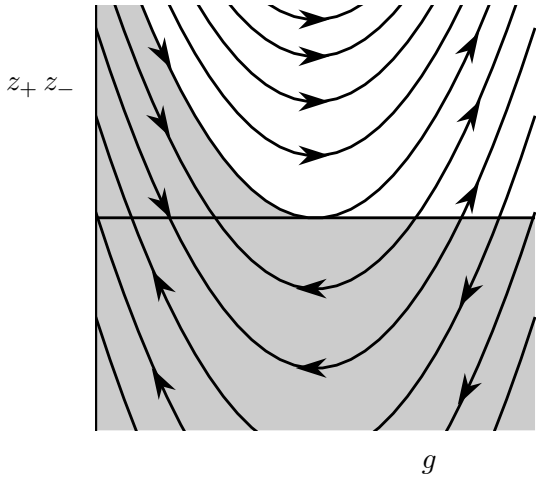


Figure 6: The renormalization flows in terms of the Gaussian coupling constant g and the product of fugacities for positive and negative unit charges $z_+\, z_-$. At large g the product is relevant, but when it starts out to be negative, it suppresses the value of g, so that eventually it becomes irrelevant. As a result in the shaded region the product $z_+\, z_-$ renormalizes eventually to zero.

As a starting point we make the following observations. Above we argued that the Potts model at non-critical temperature admits positive unit charges, but not negative. When negative unit charges are introduced, this will affect the renormalization of g. If the product of fugacities $z_+ z_- > 0$, the coupling constant g will grow, thus weakening the electric interaction by screening. However, if $z_+ z_- < 0$, the effect is opposite and g will decrease. At first $z_\pm$ will both grow because they are relevant. This will strengthen the effect on the value of g, which will decrease faster. However, this makes it unavoidable that g passes the value where the $z_{\pm 1}$ are marginal, and turn irrelevant. From there on, $z_{\pm 1}$ both will diminish, and the model returns to the Gaussian model, but with a much smaller value of g. A sketch of this RG flow in terms of g and the product $z_+\, z_-$ is given in figure 6. If we assume that the fugacity of the positive unit charges, z_+ is positive, we see that for large enough g the final direction of the RG flow depends critically on the sign of z_-. If it too is positive, it will grow indefinitely under RG, but if it is negative, it will only initially grow, and eventually diminish. This observation shows that the locus $z_- = 0$, is critical.

All this sums up to the conclusion that the Potts model (parametrized by its temperature) is a critical locus in a larger parameter space. A height model which embeds the height version of the Potts model will generally contain also the negative unit charges. Thus we propose: if a model $M(t)$, parametrized by t, in a more general parameter space, intersects with the Potts model, it is to be expected that that intersection is the critical point of M, because at that point the negative unit charges disappear.

We will now proceed to show that the O(n) model defined by (29), intersects with a Potts model at some value of the O(n) coupling constant. Consider a CPL model on a kagome lattice (see figure 7). We then use the ADE procedure to turn it into a Potts model, or a height model. We choose the Potts variables to live in the hexagons, and the neutral state in the triangles. This is corresponds to a Potts model on the triangular lattice. Likewise in the height model the heights in the hexagons are even multiples of π and in the triangles odd multiples of π.

A sequence of local height configurations around a vertex of the kagome lattice, are listed in table 2. Equivalent configurations can be obtained from these by a shift in height, or a

lattice symmetry. The configurations on the other sublattices of Kagome vertices are obtained by turning over $2\pi/3$ or $4\pi/3$, the weights being unchanged.

The weights for the CPL model and the equivalent Potts model are determined by the weight ratio of the acute and obtuse bends as in the diagrams  . Let the weight of the obtuse bends be 1 and the weight of (two) acute bends equal to B. The local weight for the height model corresponding to the Potts model, is given in the second line of table 2. When there is a local gradient in the configuration, only one of the two loop diagrams is compatible, with weight 1 or B respectively. When the heights form a saddle configuration, both loop diagrams fit, and should be summed over. We introduce the parameter $p = (S_{h+\pi}/S_h)^{1/3}$, where S is the eigenvector of the A_∞ diagram with eigenvalue $\sqrt{q} = p^3 + p^{-3}$.

The parameter B corresponds to temperature variations of the Potts model. The critical point of this Potts model is known, but unimportant in this discussion.

In the configurations permitted in this model, all the hexagonal faces take a height which differs by π from the neighboring triangular faces. This implies that two adjacent hexagons can either be equal or different in heigh by 2π. If the triangular faces could be ignored, it would precisely be equivalent to the loop model on the hexagonal lattice. Now we propose to choose weights for the height model on the kagome lattice such that the triangular heights are independent of each other, provided the configuration of the hexagonal faces is fixed. When triangle has neighboring hexagons of unequal height, its own height is fixed to sit in between that of the hexagons. When a triangle sits in between three hexagons of equal height, its own height can be above or below that of its neighbors. When the heights of the triangular faces are independent of the other triangular faces, these heights can be summed over.

In the configurations listed in table 2 the first has hexagons of unequal height, which thus fix the height of the triangles. In the following three configurations, however, the hexagons have height 2, and the triangles are free to take height 1 or 3. The choices of the left and right triangle are independent if the weight ratio for the two options of one triangle does not depend on the state of the other triangle. This is expressed in the third line of table 2, which gives a factor C for each triangle below its neighboring hexagonal neighbors, and D when it is above. For this choice of weights the summation over the triangular faces can be done trivially. It will result in a height model with hexagonal faces only, hopefully a height version of the loop model on the hexagonal lattice (19).

From the reasoning above, this latter model will be critical where it intersects with the Potts

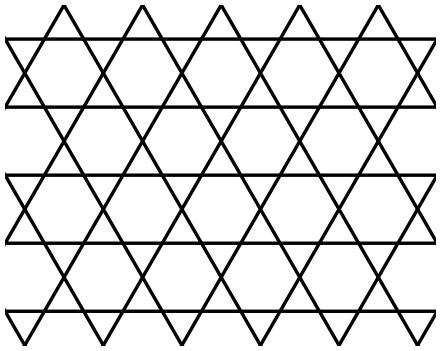


Figure 7: A section of the kagome lattice.

model. It requires some formula manipulation to see if and where the models intersect: the weights in the second and third line in the table must correspond. Since both models trace a two-dimensional manifold, and there are just three weights to match, we can expect a one-dimensional intersection. At that intersection, we then expect the O(n) model to have its phase transition.

The value of q simply follows from the ADE construction, $q = (p^3 + p^{-3})^2$, but the value of n is not immediately obvious. The summation over the heights in the triangles has to be performed first: a triangle above (below) its neighboring hexagon gives weight D (C). When a triangle sits between three hexagons of equal height, it can take its height one unit above, or one unit below these, independently of the other triangles. The weight, after summation, is then $C^3 + D^3$. The other triangles have either two hexagons above it, and one below, or the other way around, with weights C^2D or CD^2, respectively. This we will interpret as the height representation of the hexagonal loop model (19).

Exercise 4.10 *Calculate the intersection of the model and show that, at the intersection point (i) the O(n) parameters x and n in (19) and (29) are related by $x^2(2 \pm \sqrt{2-n}) = 1$, and that (ii) the Potts model is in the antiferromagnetic regime and at zero temperature, and (iii) that the relation between n and the number q of states in the Potts model is $n = 2 - (2-q)^2$.*

In summary, the two height models defined in table 2 are equivalent to the Potts and O(n) model respectively. The point where they intersect is expected to be a (the) critical point of the O(n) model. The fact that the intersection takes place in the antiferromagnetic regime of the Potts model, means that the thermal parameter of the Potts model is irrelevant, so that both models are in fact critical at the intersection.

Assertion (iii) of the exercise together with the relation between q and g, eq. (47) implies that $n = -2\cos(4\pi g)$. This is different from the accepted notation in the literature i.e. $n = -2\cos(\pi g)$. This is related to the fact that in the model just constructed, the steps across domain walls have amplitude 2π, since they correspond to two steps of the Potts model. To keep contact with most of the literature, we will rescale the heights $h_{\mathrm{O}(n)} = h_{\mathrm{Potts}}/2$, and rescale the charges $e_{\mathrm{O}(n)} = 2e_{\mathrm{Potts}}$, and the coupling constant $g_{\mathrm{O}(n)} = 4g_{\mathrm{Potts}}$ accordingly. Below the suffix O(n) will be dropped.

4.5.1 The O(n) thermal exponent

Now remember that the energy operator of the Potts model was a positive unit charge, which now becomes a positive double charge. The energy operator of the O(n) model is then a negative

config.	$\begin{matrix} & 0 & \\ 1 & & 1 \\ & 2 & \end{matrix}$	$\begin{matrix} & 2 & \\ 1 & & 3 \\ & 2 & \end{matrix}$	$\begin{matrix} & 2 & \\ 1 & & 1 \\ & 2 & \end{matrix}$	$\begin{matrix} & 2 & \\ 3 & & 3 \\ & 2 & \end{matrix}$
Potts	1	B	$p + B/p^2$	$1/p + Bp^2$
O(n)	1	CD	C^2	D^2

Table 2: A height model on the Kagome lattice. The height configurations are given in the top line with the even heights placed in the hexagons. The Boltzmann weights can be chosen so that the model is equivalent to the Potts model (second line) or O(n) model (last line), respectively.

charge of the same magnitude, that the Potts model had managed to avoid so carefully. This leads to the following (charge neutral) construction of the two-point energy function of the O(n) model

$$|r_1 - r_2|^{4/g} \int \mathrm{d}^2 r_3 \, \mathrm{d}^2 r_4 \, |r_3 - r_4|^{4/g} \prod_{i=1,2;j=3,4} |r_i - r_j|^{-4/g} \;=\; |r_1 - r_2|^{4-8/g} \,, \tag{58}$$

the result from simple power counting. This corresponds to the exponent,

$$x_\varepsilon = \frac{4}{g} - 2 \qquad \text{with} \qquad n = -2\cos(\pi g) \tag{59}$$

.

Again, like in the case of the Potts model, the exponent can be analytically continued beyond $n = 2$. The case $g \geq 1$ belongs to $x^{-2} = 2 + (2-n)^{1/2}$ and $g \leq 1$ to $x^{-2} = 2 - (2-n)^{1/2}$. But since the thermal exponent for the large-x branch is irrelevant, we conclude that the whole interval $x^{-2} < 2 + (2-n)^{1/2}$ is attracted to this fixed point, much like in the transition sheet of the Potts model, figure 4. Apparently the phase with relatively large x, the phase dense with loops, is also critical. This is precisely what one might expect from figure 6 in which the shaded regime flows eventually to the Gaussian model. The behavior is a perfect image of that at the transition manifold of the Potts model, including the value of the exponents. It turns out, that we have now established two relations between Potts and O(n) models, one in which $n = \sqrt{q}$ (transition manifold of dilute Potts model, on whole (n,x) sheet of O(n) model), and one with $n = 2 - (2-q)^2$ (O(n) critical $\leftrightarrow$ Potts AF). This explains why the thermal exponent of the O(n) model has the same expression as the crossover exponent of the tricritical Potts model (48).

The value of the background charge has changed in a non-trivial way in last the equivalence. This is because an "O(n) loop" is not simply two Potts loops with the same sign. It may be a local collaboration of Potts loops that go side by side for a while, and separate later. The background charge is determined completely by the magnitude of the step represented by a domain wall (again π after the rescaling above), and the weight of closed loops (59): $n = -2\cos(\pi g) = 2\cos(e_o)$. It is natural to choose the smallest value of e_o, but for the sign we have to consult not only the weight of the loop but separately of the domain wall encircling a lower or a higher plateau. The result is

$$e_o = g - 1 \tag{60}$$

the same expression as (46), but with opposite sign.

Exercise 4.11 *Verify that the thermal exponent 59 can be calculated also without screening charges, but in the configuration with total charge $-2e_o$.*

4.5.2 The O(n) magnetic exponent

There are many other operators that one can construct in the O(n) model, the most notable probably the spin operator itself, conjugate to the magnetic field. In this paragraph we will calculate its exponent. As before a way to approach this is first to inspect the two-point function $\langle s_o \cdot s_r \rangle$, the scalar product between two distant spin verctors. The diagrams that contribute to that, in the expansion in powers of x, are different from the ones in the partition sum: now the number of bonds incident on o and r must be odd, while in all other vertices it is even. In the height language this has an obvious interpretation: in site o and r there is a screw dislocation of one step π, i.e. a magnetic charge of strength $m = \pm\frac{1}{2}$.

At first sight one may expect an exponent equal to $gm^2/2 = g/8$, but this is not to be. The the domain walls that terminate on o or r gain a phase factor when they curve. In closed domain

walls the integrated curvature is always the same, but with these open domain walls, this is not the case. The domain walls can spiral around their terminals indefinitely, and accordingly pick up arbitrary phase factors. We will have to correct this.

Note, that as the domain walls spiral about their terminal, they bring the height of their terminal up or down, as in the case of the anyons. Consider in figure 8 the diagram spiraling around its terminal. If the height at that terminal is defined by counting the steps up and down along a fixed path from to infinity, this height is raised by with the number of windings of the path. Thus we can counteract the phase factor of the spiraling by the extra insertion of an electric charge, neutralizing the curvature effect. This is done precisely by a charge $-e_o$. It takes a little thought to conclude that the same charge is needed at the negative and the positive vortex, at either terminal of the open diagram.

As a result, the spin-spin correlation function is that of the combination of a magnetic charge $m = \frac{1}{2}$ and an electric charge $e = -e_o$, on one side and $m = \frac{-1}{2}$ with $e = -e_o$ on the other. The resulting exponent is

$$x_s = \frac{g}{8} - \frac{(g-1)}{2g} \tag{61}$$

Exercise 4.12 *It is not obvious that this electro-magnetic charge has no spin, i.e. that there is no phase factor associated with turning the position of s_r around that of s_o. Verify that this is the case.*

[b]

4.5.3 Intersections

In this whole calculation we have considered only O(n) models of which the diagrams of the high-temperature have no intersections. The O(n) spin model on arbitrary (two-dimensional) lattices or with arbitrary interaction, the paths in the diagrams do intersect (but still pick up a factor n for each closed loop. Therefore the special O(n) model defined by (29) can be considered as generic, only if the loops are irrelevant. Again we will inspect this by considering the (connected) two-point function. This is the same as the two-point function of joint terminals of 4 legs. This can be calculated completely analogously with the calculation of the magnetic exponent, and thus results in

$$x_\times = 2g - \frac{(g-1)^2}{2g} \tag{62}$$

Indeed this is irrelevant at the O(n) critical point.

Exercise 4.13 *Verify that also in this case the same charge $-e_o$ precisely cancels the total net curvature of the four legs. And verify that this exponent is irrelevant at the critical point of the O(n) model for $-2 \leq n \leq 2$.*

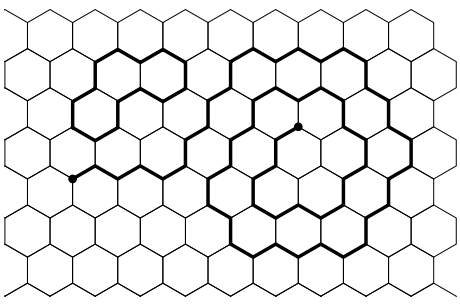


Figure 8: Diagram contributing to the spin-spin correlation function of the O(n) model.

Unfortunately, the operator is relevant in the dense phase (i.e. dense with loops), and for this reason this phase cannot serve as the low temperature phase of the generic O(n) model. This argument is discussed in [28], which proposes a model with intersecting loops to describe the low temperature phase of the O(n) spin model. This model, also called the Brauer model, was already known, and studied in [13].

4.5.4 Tricritical O(n) models

Just like in the Potts model the transition of the O(n) model can be turned discontinuous by altering the model like adding a four-spin interaction, or thermally distributed vacancies. The description of this phase transition for a long time evaded description by these methods, except for n=1, the Ising model, because it is identical to the q=2 Potts model, and for n=0 [20], see exercise 3.8. These cases could not be easily generalized to general n. Recently this problem was solved, for the details we refer to the original publication [29]. Unfortunately it suffers from the same restriction as the dense phase mentioned above: it is applicable only to those O(n) spin models that can be represented by diagrams of non-intersecting loops.

5 Summary and perspective

The applications of loop models are surprisingly numerous. Methods to study critical systems, like CG and SLE, are based upon it. The language allows for methods and investigation of operators that are not accessible in completely local theories, but make physical sense (consider e.g. the probability that two spins in the Ising model sit in the same up cluster, or that two unequal spins sit on the same domain boundary as an unequal pair elsewhere.)

We have considered models that admit a single component height representation. Several models with a more dimensional height lattice, have been studied by these methods[12, 22, 30], with remarkable results. It appears that we have only scratched the surface of what is possible.

The assertion that alternative calculations of multipoint correlation functions are the same, leads to interesting conjectures for integral expressions. In the text only one example is given, but the methods to calculate these correlation functions admit many more. A proof of these conjectures, or better, a mathematical framework in which they are obvious would be very interesting. At the moment there is not even a numerical verification.

In these lectures we have among the applications of loop models, focussed on the Potts model and the O(n) model. A number of other applications, such as percolation, the self-avoiding walk, spanning trees are special limits of these. For other applications like the hard hexagon and square models these methods also give direct results for the critical exponents, and other universal quantities.

The CG method was the first analytic calculation of numerous critical exponents in two dimensions. The results have been confirmed later by CFT and by exact solutions of integrable models. At the moment the CFT approach is capable of giving more extensive results, and SLE more rigorous. An advantage of CG that remains, is that the results and the original models and operators are directly related. In SLE and CFT this connection is usually made indirectly.

A challenging problem in these same models is the calculation of off-critical thermodynamic scaling functions, which are also universal objects. In general these are thermodynamic quantities as function of (at least) two thermodynamic parameters infinitesimally close to the critical point. This problem has not been solved analytically, and it is conceivable that the CG approach is capable of it.

References

[1] Kadanoff, L. Physics 2, 263 (1966)

[2] Niemeijer, T. and van Leeuwen, J.M.J. , Physica 71, 17 (1974) and *Phase transitions and critical phenomena* 6, p425, C.Domb ans M.S. Green eds. Academic Press (1976)

[3] Wilson, K.G., Phys. Rev. B 4, 3174 (1971)

[4] Kadanoff, L.P., J. Phys A, 11, 1399 (1978)

[5] Nienhuis, B., J. Stat. Phys. **34**, (1984) 731 and in *Phase Transitions and Critical Phenomena*, C. Domb, J.L. Lebowitz eds., Academic Press, (London 1987) p1

[6] L. Onsager, Phys. Rev. B 65, 117 (1944)

[7] E.H. Lieb, Phys. Rev. Lett. 18, 1046 (1967)

[8] R.J. Baxter, Ann. Phys. 70, 193 (1972)

[9] R.J. Baxter, P.A. Pearce, J. Phys. A 16, 2239 (1983)

[10] R.J. Baxter, J. Phys. A 13, L61 (1980)

[11] R.J. Baxter, Exactly solved models in statistical mechanics, Academic Press, New York (1982).

[12] J.L. Jacobsen, Kondev J. Nucl. Phys. B 532 , 635 (1998); J. Kondev, Jacobsen J.L. Phys. Rev. Lett. 81, 2922 (1998) ; J.L. Jacobsen, Kondev J J. Stat. Phys. 96, 21 (1999)

[13] M.J. Martins, B. Nienhuis, Rietman R. Phys. Rev. Lett. 81, 504 (1998)

[14] V. Pasquier, J. Phys. A 20, 5707 (1987) and J. Phys. A 20, L1229 (1987)

[15] E. Verlinde, Nuclear Physics B 300, 360 (1988)

[16] P. di Francesco, H. Saleur, J.-B. Zuber, J. Stat. Phys. 49, 57 (1987)

[17] S.O. Warnaar, B. Nienhuis, K.A. Seaton, Phys. Rev. Lett. 69, 710 (1992)

[18] : F.Y. Wu, Rev. Mod. Phys. 54, 235 (1982)

[19] J.L. Jacobsen, H. Saleur, Nucl. Phys. B 716, 439 (2005)

[20] B. Duplantier, H. Saleur, Phys. Rev. Lett. 59 (1987)

[21] O. Foda, B. Nienhuis, Nucl. Phys. B 324, 643 (1989)

[22] J. Kondev, Int. J. Mod. Phys. B, 11, 153 (1997)

[23] R.J. Baxter, J. Phys. A 15, 3329 (1982)

[24] B. Nienhuis, A.N. Berker, E.K. Riedel, and M. Schick, Phys. Rev. Lett. 43, 737 (1979)

[25] B. Nienhuis, Physica A 251, p104 (1998)

[26] M. Den Nijs, Phys. Rev. B. 27, 1674 (1983)

[27] B. Nienhuis, and H.J.F. Knops, Phys. Rev. B 32, 1872 (1985)

[28] J.L. Jacobsen, N. Read, H. Saleur, Phys. Rev. Lett. 90, 090601 (2003)

[29] B. Nienhuis, W.A. Guo, H.W.J. Blote, Phys. Rev. B 78, 061104 (2008)

[30] J. Kondev, de J. Gier, B. Nienhuis, J. Phys. 29, 6489 (1996)

[31] A.N. Berker and L.P. Kadanoff, J. Phys. A: Math. Gen. 13, L259-L264 (1980)

[32] J.L. Cardy, in *Phase Transitions and Critical Phenomena*, C. Domb, J.L. Lebowitz eds., Academic Press, London (1987)

[33] E. Fradkin and L.P. Kadanoff, Nucl. Phys. B 170, 1-15, (1980)