

Summer school @ Izmir 2014

On the thermodynamics and correlation functions
of lattice models solvable by Bethe ansatz

(I) General formalism

(I.1) Conventions for vectors and matrices.

$$B_{\alpha} = \{ e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_d = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \} \quad (1)$$

Canonical standard basis of \mathbb{C}^d , $\mathbb{C}^d = \langle B_{\alpha} \rangle$.

$$x \in \mathbb{C}^d \iff x = x^* e_x, x^* \in \mathbb{C}. \quad (2)$$

There is unique $e_{\beta}^* \in \text{End } \mathbb{C}^d$ with

$$e_{\beta}^* e_{\gamma} = \delta_{\beta}^{\gamma} e_{\beta}, \quad = 1. \quad e_{\beta}^* e_{\gamma}^* = \delta_{\beta}^{\gamma} e_{\beta}^*. \quad (3)$$

$$B_{\text{End } \mathbb{C}^d} = \{ e_{\beta}^* \mid \alpha, \beta = 1, \dots, d \} \text{ basis of } \text{End } \mathbb{C}^d, \quad (4)$$

$$A \in \text{End } \mathbb{C}^d = \{ A = A_{\beta}^* e_{\beta} \mid \text{"sum convention"} \} \quad (5)$$

$$B_{\text{End } (\mathbb{C}^d)^{\otimes L}} = \{ e_{\beta_1}^{x_1} \otimes \dots \otimes e_{\beta_L}^{x_L} \mid \alpha_1, \beta_1 = 1, \dots, d \} \quad (6)$$

is a basis of $\text{End } (\mathbb{C}^d)^{\otimes L}$, $A \in \text{End } (\mathbb{C}^d)^{\otimes L} \iff$

$$A = A_{\beta_1, \dots, \beta_L}^{x_1, \dots, x_L} e_{\beta_1}^{x_1} \otimes \dots \otimes e_{\beta_L}^{x_L}. \quad (7)$$

• "Local basis": Operators acting on "site j "

$$e_j^{x_{\beta}} = e_{\alpha_1}^{x_1} \otimes \dots \otimes e_{\alpha_{j-1}}^{x_{j-1}} \otimes e_{\beta}^{x_j} \otimes e_{\alpha_{j+1}}^{x_{j+1}} \otimes \dots \otimes e_{\alpha_L}^{x_L} \quad (8)$$

Then we can define operators acting non-trivially
e.g. only on sites j, k :

$$A_{jk} = A_{\beta}^{x_j} e_{\beta}^{x_k} e_{\alpha}^{x_k}. \quad (9)$$

(I.2) Yang-Baxter equation and fundamental models.

- $R(\lambda, \mu) \in \text{End}(\mathbb{C}^d \otimes \mathbb{C}^d)$, $d > 1$, solution of the Yang-Baxter equation

$$R_{12}(\lambda, \mu) R_{13}(\lambda, \nu) R_{23}(\mu, \nu) = R_{23}(\mu, \nu) R_{13}(\lambda, \nu) R_{12}(\lambda, \mu) \quad (10)$$

- R regular,

$$R(0, 0) = P$$

where P is the transposition matrix $P = e_\alpha^\beta \otimes e_\beta^\alpha$. (11)

- Then we associate a fundamental model with R , setting

$$T_a(\lambda) = R_{a1}(\lambda, \nu_1) \dots R_{an}(\lambda, \nu_n), \quad (12)$$

the monodromy matrix (space with index "a" is called auxiliary space),

$$t(\lambda) = \text{tr}_a T_a(\lambda) \quad (13)$$

the transfer matrix.

- It follows from (10) that

$$\check{R}(\lambda, \mu) (T(\lambda) \otimes T(\mu)) = (T(\mu) \otimes T(\lambda)) \check{R}(\lambda, \mu) \quad (14)$$

where $\check{R} = PR$ ("Yang-Baxter algebra"). Hence,

$$[t(\lambda), t(\mu)] = 0$$

which means that the eigenvectors of $t(\lambda)$ (15)
are independent of λ .

- If $\nu_1 = \nu_2 = \dots = \nu_L (= 0)$ the model is called homogeneous, inhomogeneous otherwise.

For the homogeneous model we have

$$\begin{aligned}
 t(0) &= \text{tr}_a P_{aL} \dots P_{a1} = \underbrace{\text{tr}_a P_{a1}}_{(11)-(13)} P_{aL} P_{aL-1} \dots P_{a2} \\
 &= \text{tr}_a e_a^\alpha e_{\alpha}^\beta = \delta_{\alpha}^{\beta} e_{\alpha}^\beta = 1 \\
 &= P_{aL} P_{aL-1} \dots P_{a2} = \hat{U}, \tag{16}
 \end{aligned}$$

shift operator, generator of the action of the cyclic group e_L on $(\mathbb{C}^d)^{\otimes L}$.

Moreover, setting $H_{ab} = \partial_\lambda R_{ab}(\lambda, 0)|_{\lambda=0} = 1$

$$\begin{aligned}
 t'(0) &= \text{tr}_a P_{aL} \underbrace{H_{aL} P_{aL-1} \dots P_{a1}}_{\frac{\delta P^L}{\delta \lambda} = 1} + \dots + \text{tr}_a P_{aL} \dots P_{a1} H_{a1} \\
 &= P_{aL-1} \dots P_{a1} H_{aL-1} = \underbrace{P_{aL} \dots P_{a1} H_{a1} P_{a1}}_{= H_{a2} P_{aL} \dots P_{a1}} \\
 (16) &= \hat{U} H_{aL-1} + \dots + \hat{U} H_{a2} + \underbrace{H_{a1} \hat{U}}_{= \hat{U} H_{a1}} \\
 &= \hat{U} \sum_{j=1}^L H_{a_{j+1} j}, \text{ where } H_{a1} := H_{aL}. \tag{17}
 \end{aligned}$$

We call H the Hamiltonian of the fundamental model. Combining (16), (17) we get

$$t(\lambda) = \hat{U}(1 + \lambda H + O(\lambda^2)). \tag{18}$$

- Since the eigenvectors of $t(\lambda)$ are independent of λ , we can obtain the eigenvalues of H from those of $t(\lambda)$ by taking the logarithmic derivative at $\lambda=0$.

(I.3) Constructing the statistical operator (density matrix of (grand-) canonical ensemble).

- We are interested in fundamental models at finite temperature. Thus, we need to know

$$S := e^{-\frac{H}{T}} \quad (19)$$

expressed in a way compatible with the integrable structure (10). This can be done in a general way if we require an additional property of the R-matrix, unitarity,

$$R_{12}(\lambda, \mu) R_{21}(\mu, \lambda) = 1 \quad . \quad (20)$$

Remarks:

(a) Unitarity is a weak additional requirement. For all models that come to my mind the R-matrix can be made unitary by multiplying entries by a factor (e.g. XYZ, Hubbard, $SU(N)$)

$$(b) (20) = ,$$

$$\partial_\lambda R_{12}(\lambda, 0) P_{21} + P_{12} \partial_\lambda R_{21}(0, \lambda) = 0$$

$$\Rightarrow H_{12} = - \partial_\lambda R_{21}(0, \lambda)|_{\lambda=0} P_{21} = - \partial_\lambda \tilde{R}_{12}(0, \lambda)|_{\lambda=0} \quad . \quad (21)$$

- We introduce N auxiliary spaces $\bar{\tau}_1, \dots, \bar{\tau}_N$ and two types of monodromy matrices associated with these,

$$T_J(\lambda) = R_{JL}(\lambda, v) \dots R_{j_1}(\lambda, v) \quad , \quad (22)$$

$$\bar{T}_J(\lambda) = R_{1J}(v, \lambda) \dots R_{JF}(v, \lambda) \quad . \quad (23)$$

$$\Rightarrow \text{tr}_J T_J(\lambda) = t(\lambda) \quad , \quad \text{tr}_J \bar{T}_J(\lambda) = \bar{t}(\lambda) \quad (24)$$

where the latter is true by definition.

- Structure of $E(\lambda)$:

Define the parity operator $\hat{P} \in \text{End}(\mathbb{C}^{\alpha})^{\otimes L}$,

$$\hat{P} e_{j,\alpha}^{\beta} \hat{P} = e_{L-j+1,\alpha}^{\beta}. \quad (25)$$

\Rightarrow (set $\nu=0$):

$$E(\lambda) = \hat{P} \hat{U} \left\{ (PRP)_{j,L}(0,\lambda) \dots (PRP)_{j,1}(0,\lambda) \right\} \hat{P} \quad (26)$$

$$\begin{aligned} \stackrel{(21)}{\Rightarrow} E(0) &= \hat{P} \hat{U} \hat{P} = \hat{P} P_0 P_1 \dots P_{L-1} \hat{P} = P_{L-1} \dots P_1 = U^{-1} \\ + P_{\nu=1}^2 & E'(0) = \hat{P} \hat{U} \left(\sum_{j=1}^L \partial_{\lambda} \underbrace{(PRP)_{j,j}(0,\lambda)}_{= R_{jj-1}}|_{\lambda=0} \right) \hat{P} \\ &= U^{-1} \sum_{j=1}^L \partial_{\lambda} R_{j,j-1}(0,\lambda)|_{\lambda=0} \stackrel{(21)}{=} -U^{-1} H = -H U^{-1} \end{aligned} \quad (27)$$

$$\Rightarrow E(\lambda) = (1 - \lambda H + O(\lambda^2)) U^{-1}. \quad (28)$$

• Using the "Trotter formula", (18), (29) =,

$$\overbrace{E(\lambda) t(-\lambda)}^{=: A(\lambda)} = 1 - 2\lambda \underbrace{H + O(\lambda)}_{=: A(\lambda)} \quad (30)$$

$$\begin{aligned} \stackrel{c_{22},}{\Rightarrow} \lim_{N \rightarrow \infty} \left(E(\frac{\beta}{N}) t(-\frac{\beta}{N}) \right)^{\frac{N}{2}} &= \lim_{N \rightarrow \infty} \left(1 - \frac{2\beta}{N} A(\frac{\beta}{N}) \right)^{\frac{N}{2}} \\ &= e^{-\beta \lim_{N \rightarrow \infty} A(\frac{\beta}{N})} = \boxed{e^{-\beta H}}. \end{aligned} \quad (31)$$

Thus, we have represented the statistical operator S as a limit of a product of transfer matrices.

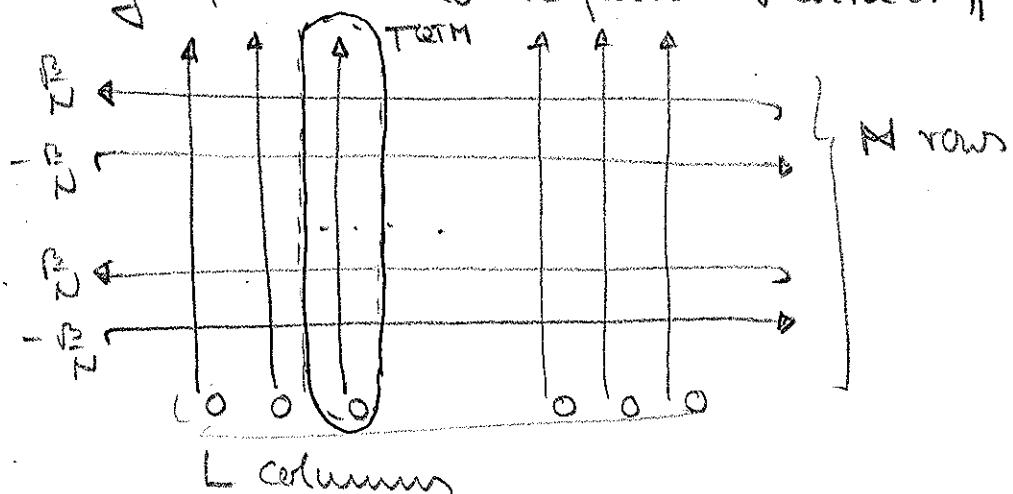
• Interpretation: "Trotter number approximant" and partition function of a special vertex model.

$$S_{N,L} := \left(E(\frac{\beta}{N}) t(-\frac{\beta}{N}) \right)^{\frac{N}{2}} \quad (32)$$

defines a special vertex model, where partition function,

$$Z_{N,L} = \text{Tr}_{1\dots L} \left(e^{(\beta_N) t} (-\beta_N)^{\frac{L}{2}} \right)^{\frac{N}{2}} \quad (33)$$

tends to the partition function $Z_L = N_{1\dots L} e^{-\beta H}$ of the fundamental lattice model with Hamiltonian. Graphically (for those who are familiar with it)



- The quantum transfer matrix and its monodromy matrix

$$\begin{aligned}
 S_{N,L} &= N_{1\dots N} \bar{T}_N(\beta_N) T_{N-1}(-\beta_{N-1}) \dots \bar{T}_2(\beta_2) T_1(-\beta_1) \\
 &= \text{Tr}_{1\dots N} \bar{T}_N(\beta_N) T_{N-1}^t(-\beta_{N-1}) \dots \bar{T}_2(\beta_2) T_1^t(-\beta_1) \\
 (22)(23) &= N_{1\dots N} R_{1N}(v, \beta_N) \dots R_{LN}(v, \beta_1) \\
 &\quad \cdot R_{11}^{t_1}(-\beta_1, v) \dots R_{NL}^{t_1}(-\beta_N, v) \dots \\
 &\quad \cdot R_{12}(v, \beta_2) \dots R_{L2}(v, \beta_2) \\
 &\quad R_{11}^{t_1}(-\beta_N, v) \dots R_{1L}^{t_1}(-\beta_1, v) |_{v=0} \\
 &\quad \text{Tr}_{1\dots N} R_{1N}(v, \beta_N) R_{N-1N}^{t_1}(-\beta_N, v) \dots R_{12}(v, \beta_2) R_{11}^{t_1}(-\beta_1, v) \\
 &\quad \dots \\
 &\quad R_{LN}(v, \beta_1) R_{N-1L}^{t_1}(-\beta_N, v) \dots R_{L2}(v, \beta_2) R_{1L}^{t_1}(-\beta_1, v) |_{v=0} \\
 &= N_{1\dots N} \bar{T}_1^{QTM}(0) \dots \bar{T}_L^{QTM}(0) \quad (34)
 \end{aligned}$$

where

$$T_j^{\text{QTM}}(\lambda) = R_{jN}(\lambda, \beta_N) R_{N-1,j}^{t_1}(-\beta_N, \lambda) \dots R_{dI}(\lambda, \beta) R_{Ij}^{t_1}(-\beta_N, \lambda) \quad (35)$$

Remark: By definition

$$R_{\frac{p+s}{2}}^{t_1 \times T} = R_{\frac{p+s}{2}}^T \quad (36)$$

• Young-Baxter algebra.

Taking the transpose with respect to space 1 in the Young-Baxter equation =,

$$R_{23}(\lambda, \mu) R_{12}^{t_1}(v, \lambda) R_{13}^{t_1}(v, \mu) = R_{13}^{t_1}(v, \mu) R_{12}^{t_1}(v, \lambda) R_{23}(\lambda, \mu) \quad (37)$$

$$\Rightarrow [R_{jk}(\lambda, \mu) T_j^{\text{QTM}}(\lambda) T_k^{\text{QTM}}(\mu) = T_k^{\text{QTM}}(\mu) T_j^{\text{QTM}}(\lambda) R_{jk}(\lambda, \mu)] \quad (38)$$

which means that T^{QTM} generates a representation of the Young-Baxter algebra (YBA).

• Summary.

We shall call N "the Trotter number". Then S_{NL} may be called a "finite Trotter number approximation" to the statistical operator S_L (of the canonical ensemble).

We have expressed S_{NL} as trace of a functional power of a certain monodromy matrix T^{QTM} , which satisfies the YBA relations (we related it to some integrable structure).

(I.4) Thermodynamics of fundamental lattice models

• QTM.

$$t^{\text{QTM}}(\lambda) = \text{tr}_j T_j^{\text{QTM}}(\lambda) \quad (39)$$

is called the "quantum transfer matrix" (QTM) of the fundamental lattice model with Hamiltonian (7) ((note: everything defined solely in terms of the R-matrix))

$$\stackrel{(38)}{\Rightarrow} [t^{QTM}(\lambda), t^{QTM}(\mu)] = 0 \quad (40)$$

commutes with t^{QTM} , hence eigenvectors independent of λ . There are cases (e.g. XXZ, Hubbard), where t^{QTM} can be diagonalized by the ABA.

Claim: (As opposed to the usual transfer matrix)

the QTM has a gap between the unique eigenvalue of largest modulus in the vicinity of $\lambda = 0$ ("the dominant eigenvalue") and the next eigenvalue, which stays finite in the "Trotter limit" $N \rightarrow \infty$ (cf. exercise 2).

• Partition function and free energy.

The partition function for the fundamental model of length L is

$$\begin{aligned} Z_L &= \text{Tr}_{1\dots L} e^{-\beta H} = \lim_{N \rightarrow \infty} \text{Tr}_{1\dots L} S_{N,L} \\ (34) \quad &= \lim_{N \rightarrow \infty} N_1 \dots N_L (t^{QTM}(0))^L = \sum_{n=0}^{\infty} \Lambda_n^L(0), \end{aligned} \quad (41)$$

where the Λ_n are the eigenvalues of the QTM. Since the spectrum of $t^{QTM}(0)$ has a gap, the partition function behaves asymptotically as $Z_L \sim \Lambda_0^L(0)$.

$$f = -T \lim_{L \rightarrow \infty} \frac{\ln Z_L}{L} = -T \ln \Lambda_0(0) \quad (42)$$

is the free energy per lattice site in the thermodynamic limit ($L \rightarrow \infty$). It is important that f is determined by a single dominant eigenvalue of the QTM.

(I.5) Temperature depending correlation functions

- We shall find expressions of the form

$$\langle X_j^{(1)} \dots X_k^{(k-j+1)} \rangle_T = \lim_{L \rightarrow \infty} \frac{\text{tr}_{1\dots L} e^{-\beta H} X_j^{(1)} \dots X_k^{(k-j+1)}}{\text{tr}_{1\dots L} e^{-\beta H}}, \quad (43)$$

where the $X_e^{(n)}$ are any local operators, correlation functions. They can be solely expressed in terms of the dominant eigenvalue and the corresponding "dominant eigenvector",

$$\begin{aligned} \langle X_j^{(1)} \dots X_k^{(k-j+1)} \rangle_T &= \lim_{L \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{\text{tr}_{1\dots L} S_{N,L} X_j^{(1)} \dots X_k^{(k-j+1)}}{\text{tr}_{1\dots L} S_{N,L}} \\ &=: z_{N,L} \\ &= \lim_{N \rightarrow \infty} \lim_{L \rightarrow \infty} \text{tr}_{\overline{1\dots N}} \text{tr}_{1\dots L} T_1^{\text{QTM}}(0) \dots T_L^{\text{QTM}}(0) X_j^{(1)} \dots X_k^{(k-j+1)} / z_{N,L} \\ &= \lim_{N,L \rightarrow \infty} \text{tr}_{\overline{1\dots N}} (T^{\text{QTM}}(0))^{L-1} \text{tr}_j \{ T_j^{\text{QTM}}(0) X_j^{(1)} \} \dots \text{tr}_k \{ T_k^{\text{QTM}}(0) X_k^{(k-j+1)} \} \\ &\quad \cdot (T^{\text{QTM}}(0))^{L-k} / z_{N,L} \\ &= \lim_{N,L \rightarrow \infty} \frac{\sum_{n=0}^{L-k+j-1} \Lambda_n^{L-k+j-1}(0) \langle \Psi_0 | \text{tr} \{ T^{\text{QTM}}(0) X^{(1)} \} \dots \text{tr} \{ T^{\text{QTM}}(0) X^{(k-j+1)} \} | \Psi_0 \rangle}{\sum_{n=0}^{L-1} \Lambda_n^{L-k+j-1}} \\ &= \lim_{N \rightarrow \infty} \frac{\langle \Psi_0 | \text{tr} \{ T^{\text{QTM}}(0) X^{(1)} \} \dots \text{tr} \{ T^{\text{QTM}}(0) X^{(k-j+1)} \} | \Psi_0 \rangle}{\Lambda_0^{L-k+j-1}(0)} \end{aligned} \quad (44)$$

Remark: (44) was the starting point of our work on correlation functions of integrable models (FG, A Klümper, A Sei, J. Phys. A 37 (2004) 7625). The important point is that a single eigenvector contains all information about the static correlation functions at any temperature.

• (Reduced) density matrix,

Set $j=1, k=m$, $X^{(e)} = e^{\alpha_e} \beta_e$ and suppress superscript "QTM" from now on. $=$

$$D_{\beta_1 \dots \beta_m}^{\alpha_1 \dots \alpha_m} = \langle e_{1\beta_1}^{\alpha_1} \dots e_{m\beta_m}^{\alpha_m} \rangle = \frac{\langle \Psi_0 | T_{\beta_1}^{\alpha_1}(0) \dots T_{\beta_m}^{\alpha_m} | \Psi_0 \rangle}{\Lambda_0^m(0)} \quad (45)$$

define the matrix elements of

\check{V} a finite Trotter number approximant to the reduced density matrix (short "density matrix"). The thermal expectation value of any operator A being nontrivially at most on lattice sites $1, \dots, m$ is

$$\langle A \rangle_T = \lim_{N \rightarrow \infty} \text{Tr}_{\check{V}} \{ A D(T) \}. \quad (46)$$

Eq. (45) suggest the following definition,

$$D(\xi_1, \dots, \xi_m | T) = \frac{\langle 4_0 | T(\xi_1) \otimes \dots \otimes T(\xi_m) | 4_0 \rangle}{\langle 4_0 | \prod_{j=1}^m t(\xi_j) | 4_0 \rangle} \quad (47)$$

which makes sense for unnormalized eigenvectors and which we call the inhomogeneous density matrix.
Note that $D(0, \dots, 0 | T) = D(T)$.

Remarks:

- (i) Using ABA techniques $D(\xi_1, \dots, \xi_m | T)$ in the Trotter limit can be expressed as a multiple integral (F.G. Al-Klini, A. Seel, J. Phys. A 38 (2005) 1833).
- (ii) Multiple integrals factorize into products of single integrals (H. Boos, F.G. Al-Klini, J. Suzuki, JSTAT, P04001 (2006); general theorem: M. Jimbo, T. Miwa, F. Smirnov, J. Phys. A 42 (2009) 304018) and can be evaluated at short distances.

(I.6) Fermi factor expansion and asymptotics of two-point functions

For $X, Y \in \mathbb{C}^d$ define

$$X(\xi) = \text{Tr } T(\xi) X, \quad Y(\xi) = \text{Tr } T(\xi) Y \quad (48)$$

$$\langle X_i Y_{i+1} \rangle_T = \text{Tr}_{1\dots m+1} D(T) X_i Y_{i+1} \quad (49)$$

It can be represented as

$$\begin{aligned}
\langle X_1 Y_{n+1} \rangle_T &= \frac{\langle \psi_0 | X(0) t^{n+1}(0) Y(0) | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle \Lambda_0^{n+1}(0)} \\
&= \sum_{k=0}^{d^n-1} \frac{\Lambda_k^{n+1}(0) \langle \psi_0 | X(0) | \psi_k \rangle \langle \psi_k | Y(0) | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle \langle \psi_k | \psi_k \rangle \Lambda_0^{n+1}(0)} \\
&= \sum_{k=0}^{d^n-1} \frac{\langle \psi_0 | X(0) | \psi_k \rangle}{\langle \psi_k | \psi_k \rangle \Lambda_0(0)} \cdot \frac{\langle \psi_k | Y(0) | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle \Lambda_k(0)} \cdot \left(\frac{\Lambda_k(0)}{\Lambda_0(0)} \right)^n \\
&=: \bar{F}_X(k) \quad =: \bar{F}_Y(k) \quad \text{form factors of QTM}
\end{aligned} \tag{50}$$

$$\begin{aligned}
\Rightarrow \boxed{\langle X_1 Y_{n+1} \rangle_T - \langle X_1 \rangle_T \langle Y_{n+1} \rangle_T} \\
&= \sum_{k=1}^{d^n-1} \bar{F}_X(k) \bar{F}_Y(k) \left(\frac{\Lambda_k(0)}{\Lambda_0(0)} \right)^n
\end{aligned} \tag{51}$$

This is the form factor expansion of the two-point function. Since $\Lambda_0 > \Lambda_k \forall k \in \mathbb{Z}_+$, every term decays exponentially with n . \Rightarrow The next-to-leading eigenvalues $\Lambda_1, \dots, \Lambda_d$ (of degeneracy of $K \times 1$) determine the asymptotics of the two-point functions.

$$\begin{aligned}
\left(\frac{\Lambda_k(0)}{\Lambda_0(0)} \right)^n &= \exp \left\{ - \underbrace{\ln \left(\frac{\Lambda_0(0)}{\Lambda_k(0)} \right)}_{\equiv k} \cdot n \right\}, \\
&=: \xi_k^{-1}
\end{aligned} \tag{52}$$

ξ_k is called the k -th correlation length in the asymptotic expansion (51).

Remark: For the XXZ chain the correlation lengths and the amplitudes can be calculated in the Trotter limit.

ξ_k : A. Klümper, Z. Phys. B 51 (1983) 507.

\bar{F}_X, \bar{F}_Y : M. Dujarrí, F. A. K. Korłucki, JSTAT P07010 (2013), ibid P04012 (2014). (11)

(II) Calculation of some temperature-related quantities for the XXz chain

(II.1) Diagonalization of the QTM by ABA

Recall that $t(\lambda) = \text{tr}_a T_a(\lambda)$,

$$T_a(\lambda) = R_{\alpha}(\lambda - \beta_N^a) R_{\alpha a}^{t_1}(-\beta_N^a - \lambda) \dots R_{\alpha}(\lambda - \beta_N^1) R_{1 a}^{t_1}(-\beta_N^1 - \lambda) \quad (53)$$

and that (Edel's lecture)

$$R(\lambda) = \begin{vmatrix} 1 & b(\lambda) & c(\lambda) \\ c(\lambda) & b(\lambda) & 1 \\ 1 & 1 & 1 \end{vmatrix}, \quad b(\lambda) = \frac{\sinh(\lambda)}{\sinh(\lambda + y)}, \quad c(\lambda) = \frac{\sinh(y)}{\sinh(\lambda + y)} \quad (54)$$

$$\Rightarrow R_{\alpha j}(\lambda - \beta_N^a) = \begin{vmatrix} e_{j+}^+ + b(\lambda - \beta_N^a)e_{j-}^- & c(\lambda - \beta_N^a)e_{j+}^- \\ c(\lambda - \beta_N^a)e_{j-}^+ & b(\lambda - \beta_N^a)e_{j+}^+ + e_{j-}^- \end{vmatrix}, \quad (55)$$

$$R_{j\alpha}^{t_1}(-\beta_N^a - \lambda) = \begin{vmatrix} e_{j+}^+ + b(-\beta_N^a - \lambda)e_{j-}^- & c(-\beta_N^a - \lambda)e_{j+}^- \\ c(-\beta_N^a - \lambda)e_{j-}^+ & b(-\beta_N^a - \lambda)e_{j+}^+ + e_{j-}^- \end{vmatrix}. \quad (56)$$

Requirement for ABA: \exists "pseudo vacuum" $|0\rangle$

i.e. $T_+^+(\lambda)|0\rangle = 0$, $T_+^+(\lambda)|0\rangle = a(\lambda)|0\rangle$, $T_-^-(\lambda)|0\rangle = d(\lambda)|0\rangle$. Then

$$R_{\alpha j}(\lambda - \beta_N^a)e_{j+} = \begin{vmatrix} 1 & c(\lambda - \beta_N^a)e_{j-}^+ \\ 0 & b(\lambda - \beta_N^a) \end{vmatrix} e_{j+}, \quad (57)$$

$$R_{j\alpha}^{t_1}(-\beta_N^a - \lambda)e_{j-} = \begin{vmatrix} b(-\beta_N^a - \lambda) & c(-\beta_N^a - \lambda) \\ 0 & 1 \end{vmatrix} e_{j-}. \quad (58)$$

$$\Rightarrow |0\rangle = e_1 - e_2 + \dots - e_{L-1} + e_L, \quad (59)$$

$$a(\lambda) = b(-\beta_N^a - \lambda)^{\frac{N}{2}} = \left(\frac{\sinh(\lambda + \beta_N^a)}{\sinh(\lambda + \beta_N^a - y)} \right)^{\frac{N}{2}}, \quad (60)$$

$$d(\lambda) = b(\lambda - \beta_N)^M = \left(\frac{\sin(\lambda - \beta_N)}{\sin(\lambda - \beta_N + y)} \right)^M, \quad (61)$$

Thus

$$\Lambda(\lambda) = a(\lambda) \prod_{j=1}^M \frac{\sin(\lambda - \lambda_j - y)}{\sin(\lambda - \lambda_j)} + d(\lambda) \prod_{j=1}^M \frac{\sin(\lambda - \lambda_j + y)}{\sin(\lambda - \lambda_j)} \quad (62)$$

is an eigenvalue of the QTM if the Bethe eqns.

$$\frac{d(\lambda_j)}{a(\lambda_j)} \prod_{\substack{k=1 \\ k \neq j}}^M \frac{\sin(\lambda_j - \lambda_k + y)}{\sin(\lambda_j - \lambda_k - y)} = 1, \quad j = 1, \dots, M, \quad (63)$$

are satisfied for a and d according to eqn. (60), (61).

- Including a magnetic field in z -direction.

$$\Theta := \begin{vmatrix} e^\Theta & 0 \\ 0 & e^{-\Theta} \end{vmatrix} \Rightarrow \Theta \otimes \Theta = \begin{vmatrix} e^{2\Theta} & & & \\ & 1 & & \\ & & 1 & \\ & & & e^{-2\Theta} \end{vmatrix} \quad (64)$$

and, since $\check{R}(\lambda)$ is of block form ($\check{R}(\lambda) = \begin{vmatrix} * & ** & \\ * & * & * \\ * & * & * \end{vmatrix}$)

$$\check{R}(\lambda - \mu)(\Theta \otimes \Theta) = (\Theta \otimes \Theta) \check{R}(\lambda - \mu) \quad (65)$$

i.e., Θ is a spectral parameter independent representation of the TBA. $\Rightarrow \Theta T(\lambda)$ is a representation of the YBA. Now

$$e^{-(\beta H - \hbar S^z/T)} = \lim_{N \rightarrow \infty} e^{\frac{i\pi \hbar t}{T} S_{N,L}}, \quad (66)$$

$$e^{\frac{i\pi \hbar t}{T} S_{N,L}} = N_{T-L} \left\{ \left(\begin{matrix} e^{\frac{i\pi \hbar t}{T}} & 0 \\ 0 & e^{-\frac{i\pi \hbar t}{T}} \end{matrix} \right) T_1(0) \cdots \left(\begin{matrix} e^{\frac{i\pi \hbar t}{T}} & 0 \\ 0 & e^{-\frac{i\pi \hbar t}{T}} \end{matrix} \right) T_L(0) \right\}. \quad (67)$$

Thus, an external magnetic field \hbar can be included by replacing $T(\lambda) \rightarrow \Theta T(\lambda)$ with $\Theta = \frac{\hbar}{2T}$. Since $|0\rangle$, eq. (59) is still a pseudo vacuum this leads to the replacement

$$a(\lambda) \rightarrow e^{\frac{i\pi \hbar t}{T}} a(\lambda), \quad d(\lambda) \rightarrow e^{-\frac{i\pi \hbar t}{T}} d(\lambda) \quad (68) \quad (13)$$

in the Bethe eqs. and in the expression for the eigenvalue (62).⁽⁶³⁾

(II.2) Free energy in the Trotter limit (critical case)

- We want to calculate the dominant eigenvalue λ_0 in the Trotter limit. Let us introduce

$$Q(\lambda) = \prod_{j=1}^M \sin(\lambda - \gamma_j). \quad (69)$$

With this and 681 the equation (62) for the dominant eigenvalue turns into

$$\lambda_0(\lambda) Q(\lambda) = e^{\frac{i\pi}{4} d(\lambda)} Q(\lambda - y) + e^{-\frac{i\pi}{4} d(\lambda)} Q(\lambda + y). \quad (70)$$

- The γ_j parametrizing Q follow from the Bethe eqs. (63). We first have to find out which of the many solutions of (63) belongs to the dominant state. This can be guessed from looking easier ($T=0, T=\infty, \Delta=0$) and from the numerical study of systems with small Trotter number. From this we obtain the following

Claim: Let $\text{Re } y = 0$ ("critical case") and

$$S = \{z \in \mathbb{C} \mid |Im z| < |y|\}. \quad (71)$$

Then the dominant state is characterized by the unique solution of (63) for which

$$\begin{aligned} \text{(i)} \quad M &= N/2 \\ \text{(ii)} \quad \gamma_j \in S \quad \text{for } j = 1, \dots, N/2 \\ \text{(iii)} \quad \lambda_0(\lambda) &\neq 0 \quad \forall \lambda \in S \end{aligned} \quad \left. \right\} \quad (72)$$

- Define "the auxiliary function" or of the dominant state by

$$\alpha(\lambda) = \frac{e^{-\frac{i\pi}{4} d(\lambda)} Q(\lambda - y)}{a(\lambda) Q(\lambda + y)}. \quad (73)$$

$$\stackrel{?}{=} (70) \quad \Lambda_0(\lambda) Q(\lambda) = e^{-\frac{i\pi}{4}} a(\lambda) Q(\lambda - \eta) (1 + o_r(\lambda)) \quad (74)$$

(40), (69), (69)
 $\stackrel{?}{=} (70)$ 1 + o_r is holomorphic inside S where V are the Bethe roots $\lambda_j, j=1, \dots, N/2$, and its only pole is an $N/2$ -fold pole at $\lambda = -\beta/N$.

$$\Rightarrow \partial_\lambda \ln(1 + o_r(\lambda)) = \sum_{j=1}^{N/2} \frac{1}{\lambda - \lambda_j} + \frac{N}{2} \frac{1}{\lambda + \frac{\beta}{N}} \quad (75)$$

is holomorphic inside S . The same is true for the function $\ln\left(\frac{\sinh(\lambda - \mu + \eta)}{\sinh(\lambda - \mu - \eta)}\right)$ if $\lambda, \mu \in S$. Let C be a simple closed contour inside S enclosing all Bethe root and the $N/2$ -fold pole.

$$\begin{aligned} & \int_C \frac{d\mu}{2\pi i} \ln\left(\frac{\sinh(\lambda - \mu + \eta)}{\sinh(\lambda - \mu - \eta)}\right) d\mu \ln(1 + o_r(\mu)) \\ &= \ln\left(\prod_{j=1}^{N/2} \frac{\sinh(\lambda - \lambda_j + \eta)}{\sinh(\lambda - \lambda_j - \eta)}\right) - \frac{N}{2} \ln\left(\frac{\sinh(\lambda + \frac{\beta}{N} + \eta)}{\sinh(\lambda + \frac{\beta}{N} - \eta)}\right) \\ &= \ln\left(\frac{Q(\lambda + \eta)}{Q(\lambda - \eta)}\right) + \ln\left(e^{-\frac{i\pi}{4}} \frac{d(\lambda)}{a(\lambda)}\right) + \frac{i\pi}{4} \\ & \quad + \frac{N}{2} \ln \left\{ \frac{\sinh(\lambda + \frac{\beta}{N} - \eta)}{\sinh(\lambda + \frac{\beta}{N} + \eta)} \cdot \frac{\sinh(\lambda + \frac{\beta}{N})}{\sinh(\lambda - \frac{\beta}{N} - \eta)} \cdot \frac{\sinh(\lambda - \frac{\beta}{N} + \eta)}{\sinh(\lambda - \frac{\beta}{N})} \right\} \\ & \quad \xrightarrow{\text{=: } \beta e_N(\lambda)} \end{aligned}$$

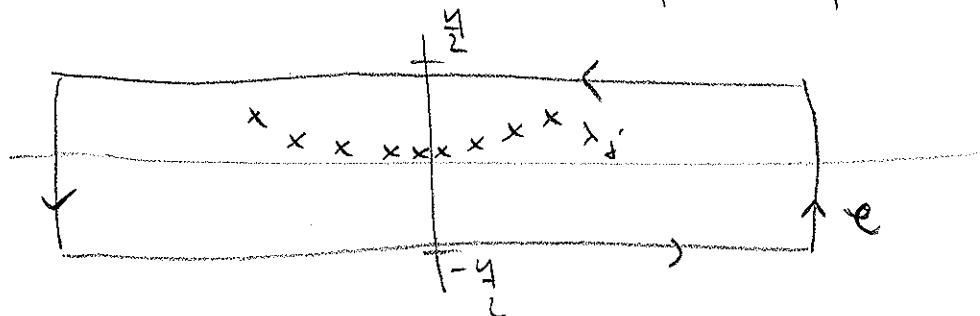
$$(73) = \ln o_r(\lambda) + \frac{i\pi}{4} + \beta e_N(\lambda)$$

partial,
no boundary
term,
see (65)

$$\begin{aligned} & \int_C \frac{d\mu}{2\pi i} \left(\partial_\lambda \ln\left(\frac{\sinh(\lambda - \mu + \eta)}{\sinh(\lambda - \mu - \eta)}\right) \right) \ln(1 + o_r(\mu)) \\ &= \operatorname{ctn}(\lambda - \mu + \eta) - \operatorname{ctn}(\lambda - \mu - \eta) =: -K(\lambda - \mu) \end{aligned} \quad (76)$$

$$\Rightarrow \boxed{\ln o_r(\lambda) = -\frac{i\pi}{4} - \beta e_N(\lambda) - \int_C \frac{d\mu}{2\pi i} K(\lambda - \mu) \ln(1 + o_r(\mu))} \quad (77)$$

This is a non-linear integral equation (NLIE) that determines the auxiliary function for the dominant eigenvalue λ_0 uniquely. The integration contour must contain all Bethe roots but no other zeros of $1+\alpha_r$. It can be chosen as follows,



• Trotter limit.

The only explicit N -dependence is in e_N . Sending N to infinity we obtain

$$\begin{aligned} \lim_{N \rightarrow \infty} e_N(\lambda) &= \lim_{N \rightarrow \infty} \frac{N}{2\beta} \ln \left\{ \frac{\sinh(\lambda + \beta) \sinh(\lambda + y - \frac{\beta}{N})}{\sinh(\lambda - \frac{\beta}{N}) \sinh(\lambda + y + \frac{\beta}{N})} \right\} \\ &= \coth(\lambda) - \coth(\lambda + y) = :e(\lambda). \end{aligned} \quad (48)$$

In the Trotter limit e is determined by the NLIE.

$$\ln \alpha_r(\lambda) = -\frac{y}{\beta} - \beta e(\lambda) - \int_{\mathbb{R}^+} d\mu K(\lambda - \mu) \ln(1 + \alpha_r(\mu)). \quad (49)$$

• Free energy.

Consider the integral

$$\int_{\mathbb{R}} d\mu \frac{1}{2\pi i} E(\mu - \lambda) \ln(1 + \alpha_r(\mu))$$

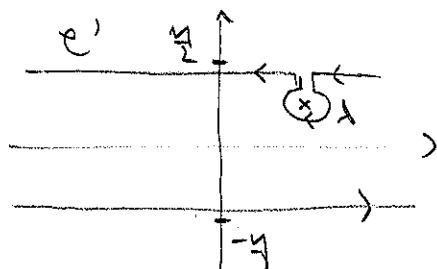
$$= \ln(1 + \alpha_r(\lambda))$$

$$+ \int_{\mathbb{R}} d\mu \left(\partial_\mu \ln \left(\frac{\sinh(\mu - \lambda)}{\sinh(\mu - \lambda + y)} \right) \right) \ln(1 + \alpha_r(\mu))$$

$$\text{partial} = - \int_{\mathbb{R}} d\mu \frac{1}{2\pi i} \ln \left(\frac{\sinh(\mu - \lambda)}{\sinh(\mu - \lambda + y)} \right) \partial_\mu \ln(1 + \alpha_r(\mu))$$

$$= \ln \left(\frac{Q(\lambda - y)}{Q(\lambda)} \right) + \sum_{n=1}^N \ln \left(\frac{\sinh(\lambda + \frac{\beta_n}{N})}{\sinh(\lambda + \beta_N - y)} \right)$$

$$= \ln a(\lambda)$$



$e - e'$ = small positive
circle around λ

$$\downarrow = \ln \left(\frac{(1+\alpha(\lambda)) Q(\lambda-\eta) \alpha(\lambda)}{Q(\lambda)} \right)^{(4)} = -\frac{h}{2T} + \ln \Lambda_0(\lambda) \quad (80)$$

$$\Leftrightarrow \boxed{\ln \Lambda_0(\lambda) = \frac{h}{2T} + \int \frac{dx}{2\pi i} e^{(x-\lambda)} \ln(1+\alpha(x))} \quad (81)$$

This is the equation for the dominant eigenvalue. Since it has no explicit N-dependence, it remains valid in the Trotter limit.

$$\Rightarrow \boxed{f(T, h) = -\frac{h}{2} - T \int \frac{dx}{2\pi i} e(x) \ln(1+\alpha(x))} \quad (82)$$

is the free energy per lattice site of the XXZ chain.

Remark:

Eqs. (81), (82) are very efficient numerically and can be treated analytically for small and large T.

(II.3) A method how to calculate correlation functions
- example: the one-point function.

The only non-trivial one-point function for XXZ is the magnetization $m(T, h) = \frac{1}{2} \langle \sigma^z \rangle$. Using (44) we have to calculate

$$\begin{aligned} \frac{1}{\Lambda_0(\xi)} \langle 4_0 | \left(\frac{A(\xi) - D(\xi)}{2} \right) | 4_0 \rangle &= \frac{1}{\Lambda_0(\xi)} \langle 4_0 | \left(A(\xi) - \frac{A(\xi) + D(\xi)}{2} \right) | 4_0 \rangle \\ &= -\frac{1}{2} + \frac{\langle 4_0 | A(\xi) | 4_0 \rangle}{\Lambda_0(\xi)} = m_N(T, h) \end{aligned} \quad (83)$$

In the Trotter limit $N \rightarrow \infty$. Within the ABA description of Bethe states this becomes ($M = N/L$)

$$m_N(T, h) = -\frac{1}{2} + \frac{\langle 0 | C(\lambda_1) \dots C(\lambda_M) A(\xi) B(\lambda_M) \dots B(\lambda_1) | 0 \rangle}{\langle 0 | C(\lambda_1) \dots C(\lambda_M) B(\lambda_M) \dots B(\lambda_1) | 0 \rangle \Lambda_0(\xi)} \quad (84)$$

This can be calculated by combining two formulae from ABA,

- First formula: „Left action formula“ (Eckles lecture)
(simple to prove)

$$\left[\prod_{j=1}^m C(\lambda_j) \right] A(\lambda_{m+1}) = \sum_{j=1}^{m+1} A(\lambda_j) C(\lambda_{m+1}, \lambda_j) \left[\prod_{\substack{k=1 \\ k \neq j}}^{m+1} \frac{1}{b(\lambda_k, \lambda_j)} \right] \prod_{k=1, k \neq j}^{m+1} C(\lambda_k), \quad (85)$$

Since $\langle 0 | D(\xi) = d(\xi) | 0 \rangle$, this can be reduced to a sum over terms of the form $\langle 0 | C(\mu_1) \dots C(\mu_m) B(\lambda_1) \dots B(\lambda_m) | 0 \rangle$ which can be calculated by means of

- Second formula: „Slavnov formula“
(hard to prove, Slavnov 89)

$$\begin{aligned} & \langle 0 | \left(\prod_{j=1}^m C(\mu_j) \right) \left(\prod_{j=1}^m B(\lambda_j) \right) | 0 \rangle \\ &= \frac{\left[\prod_{j=1}^m a(\mu_j) d(\lambda_j) \right] \left[\prod_{j,k=1}^m \operatorname{sh}(\lambda_j - \mu_k + y) \right]}{\prod_{1 \leq j < k \leq m} \operatorname{sh}(\lambda_j - \lambda_k) \operatorname{sh}(\mu_k - \mu_j)} \cdot \det((\vec{x}(\mu_1), \dots, \vec{x}(\mu_m)) \quad (86) \end{aligned}$$

where

$$x^\pm(\mu) = e(\lambda_j - \mu) - e(\mu - \lambda_j) \sigma(\mu) \quad (87)$$

and where the λ_j are supposed to satisfy the Bethe equations $1 + \sigma(\lambda_j) = 0$.

- Using (86) we obtain

$$\begin{aligned} & \frac{\langle 0 | \left(\prod_{k=1}^m C(\mu_k) \right) \left(\prod_{j=1}^m B(\mu_j) \right) | 0 \rangle}{\langle 0 | \left(\prod_{k=1}^m C(\lambda_k) \right) \left(\prod_{j=1}^m B(\lambda_j) \right) | 0 \rangle} \Lambda_b(\xi) \\ &= - \frac{1}{a(\lambda_1)(1 + \sigma(\xi))} \cdot \frac{b(\xi, \lambda_1)}{C(\xi, \lambda_1)} \left[\prod_{\substack{k=1 \\ k \neq 1}}^m b(\lambda_k, \lambda_1) \right] \\ & \quad \cdot \frac{\det_M((\vec{x}(\lambda_1), \dots, \vec{x}(\lambda_{j-1}), \vec{x}(\xi), \vec{x}(\lambda_{j+1}), \dots, \vec{x}(\lambda_m)))}{\det_M((\vec{x}(\lambda_1), \dots, \vec{x}(\lambda_M)))} \quad (88) \\ &= :g^j \cdot (1 + \sigma(\xi)) \quad \equiv :S \end{aligned}$$

$$\begin{aligned}
 \stackrel{(84), (85)}{\Rightarrow} m_n(T, h) &= -\frac{1}{2} - \sum_{j=1}^M g_j + \frac{a(\xi)}{\lambda(\xi)} \cdot C(\xi, \xi) \left[\prod_{k=1}^M \frac{1}{1 + \alpha(\lambda_k, \xi)} \right] \\
 &= \frac{Q(\xi - \eta)}{Q(\xi)} \\
 &= -\frac{1}{2} + \frac{1}{1 + \alpha(\xi)} - \sum_{j=1}^M g_j. \tag{89}
 \end{aligned}$$

- Finally we have to calculate the sum in (89).
Using that $S^{-1}\vec{x}(\lambda_j) = e_j$ we obtain

$$\begin{aligned}
 (1 + \alpha(\xi)) g_j &= \det_M ((e_1, \dots, e_{j-1}, S^{-1}\vec{x}(\xi), e_{j+1}, \dots, e_M)) \\
 &= (S^{-1}\vec{x}(\xi))^j,
 \end{aligned}$$

$$\Leftrightarrow S \vec{g} = \frac{\vec{x}(\xi)}{1 + \alpha(\xi)}. \tag{90}$$

Now,

$$\begin{aligned}
 x^j(\lambda_k) &= \lim_{\mu \rightarrow \lambda_k} \frac{e(\lambda_j - \mu) + e(\mu - \lambda_j) - e(\mu - \lambda_j)(1 + \alpha(\mu))}{\lambda_j - \mu} \\
 &= K(\lambda_j - \lambda_k) \sim \alpha'(\lambda_k)(\mu - \lambda_k) + \dots \\
 &= K(\lambda_j - \lambda_k) - \delta_{jk}^1 \alpha'(\lambda_j)
 \end{aligned} \tag{91}$$

Thus, (90) is equivalent to

$$-g_j \alpha'(\lambda_j) + \sum_{k=1}^M K(\lambda_j - \lambda_k) g_k = \frac{e(\lambda_j - \xi)}{1 + \alpha(\xi)} - \frac{e(\xi - \lambda_j)}{1 + \alpha'(\xi)}. \tag{92}$$

We transform this into a linear integral equation.
For this purpose let

$$h(\lambda, \xi) = \frac{e(\xi - \lambda)}{1 + \alpha(\xi)} - \frac{e(\lambda - \xi)}{1 + \alpha(\xi)} + \sum_{k=1}^M K(\lambda - \lambda_k) g_k. \tag{93}$$

Assume that ξ is inside our contour Γ (above eqn. (78)), say close to 0. Then h is meromorphic as a

function of λ and its only singularity inside \mathcal{C} is a simple pole with residue

$$-\frac{1}{1+\alpha(\xi)} - \frac{1}{1+\alpha^{-1}(\xi)} = -1 \quad . \quad (94)$$

Furthermore, (83) =,

$$G(\lambda_j, \xi) = g \cdot \alpha'(\lambda_j) \quad . \quad (85)$$

$$\begin{aligned} \stackrel{=}{(83)(85)} G(\lambda, \xi) &= \frac{e(\xi - \lambda)}{1 + \alpha(\xi)} - \frac{e(\lambda - \xi)}{1 + \alpha(\xi)} + \sum_{k=1}^M K(\lambda - \lambda_k) \frac{G(\lambda_k, \xi)}{\alpha'(\lambda_k)} \\ &\stackrel{\text{def}}{=} \frac{\int d\mu}{2\pi i(1+\alpha(\mu))} \cdot K(\lambda - \mu) G(\mu, \xi) \\ &\quad - \underbrace{\left(\sum_{\mu=\xi} \alpha(\mu) G(\mu, \xi) \right) \cdot \frac{K(\lambda - \xi)}{1 + \alpha(\xi)}}_{= \frac{e(\lambda - \xi) + e(\xi - \lambda)}{1 + \alpha(\xi)}} \\ (84) &= \frac{e(\xi - \lambda) + \int \frac{d\mu}{2\pi i(1+\alpha(\mu))} K(\lambda - \mu) G(\mu, \xi)}{1 + \alpha(\xi)} \quad . \quad (86) \end{aligned}$$

This is a linear integral equation for the function G . This function (and its generalizations) are important in the theory of correlation functions of the XXZ chain.

$$\begin{aligned} \stackrel{=}{(83)(85)} M(\mu, \mu) &= -\frac{1}{2} + \frac{1}{1 + \alpha(\xi)} - \sum_{j=1}^M \frac{G(\lambda_j, \xi)}{\alpha'(\lambda_j)} \\ &= \int \frac{d\mu}{2\pi i(1+\alpha(\mu))} G(\mu, \xi) + \frac{1}{1 + \alpha(\xi)} \\ &= -\frac{1}{2} - \int \frac{d\mu}{2\pi i(1+\alpha(\mu))} G(\mu, \xi) \quad . \quad (87) \end{aligned}$$

For the Trotter limit $N \rightarrow \infty$ is already taken and

is hidden in the definition of or.

- The procedure just store generators to arbitrary matrix elements of the reduced density matrix of the $\hat{X}\hat{X}^\dagger$ density which this way is represented as a multiple integral (FGr, NP Hasenfratz, A.Sel, JSTAT, P10015 (2005)).
- tell them more about correlation functions

25.8.14 THE END